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Partial Differential Equations/Calculus of Variations

On the exact controllability of a system of mixed order with essential spectrum

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Abstract

We consider the exact boundary controllability problem for the system $\boldsymbol{y}'' + \mathbf{A}_c\boldsymbol{y} = 0$ where $\mathbf{A}_c$ is a self-adjoint operator in $\mathbf{H}$ whose (non empty) essential spectrum is $\{0\}$, which implies the non-exact controllability. We construct the HUM control for all initial condition in the subspace $\mathbf{H}_c^{\perp} \times \widetilde{\mathbf{V}}_0^{\perp}$ of $\mathbf{H} \times \mathbf{V}'$. Each subspace $\mathbf{H}_c^{\perp}$ and $\widetilde{\mathbf{V}}_0^{\perp}$ is spanned by the eigenvectors associated to the eigenvalues of $\mathbf{A}_c$ satisfying the *separation condition* (Σ), and this property seems general. We also study the behavior of the control when c goes to zero. ***To cite this article: F. Ammar-Khodja et al., C. R. Acad. Sci. Paris, Ser. I 346 (2008).***

Résumé

Sur la contrôlabilité exacte d'un système d'ordre mixte ayant un spectre essentiel. On s'intéresse à la contrôlabilité exacte frontière d'un système du type $\boldsymbol{y}'' + \mathbf{A}_c\boldsymbol{y} = 0$ où $\mathbf{A}_c$ est un opérateur dans l'espace $\mathbf{H}$ dont le spectre essentiel (non vide) est $\{0\}$, ce qui implique la non contrôlabilité exacte. On construit le contrôle HUM pour toute donnée initiale dans le sous-espace $\mathbf{H}_c^{\perp} \times \widetilde{\mathbf{V}}_0^{\perp}$ de $\mathbf{H} \times \mathbf{V}'$. Les sous-espaces $\mathbf{H}_c^{\perp}$ et $\widetilde{\mathbf{V}}_0^{\perp}$ sont engendrés par les vecteurs propres associés aux valeurs propres de $\mathbf{A}_c$ satisfaisant la *condition de séparation* (Σ), propriété qui semble générale. Nous étudions également le comportement du contrôle lorsque c tend vers zéro. ***Pour citer cet article : F. Ammar-Khodja et al., C. R. Acad. Sci. Paris, Ser. I 346 (2008).***

Version française abrégée

Dans [4] on a étudié la contrôlabilité exacte de (4) et on a montré l'existence des données initiales $(\mathbf{u}^0, \mathbf{u}^1)$ qui ne sont pas exactement contrôlables. Dans le but de *caractériser le sous-espace des données initiales qui sont exactement contrôlables*, on s'intéresse dans cette Note à un exemple particulier 1D. Soient $c \in \mathbb{R}^+$ et $\omega = (0, 1)$; on veut déterminer les données initiales $(\boldsymbol{y}^{\mathbf{0}}, \boldsymbol{y}^{\mathbf{1}}) \in \mathbf{H} \times \mathbf{V}'$ où $\mathbf{H} = L^2(\omega) \times L^2(\omega)$ et $\mathbf{V} = H_0^1(\omega) \times L^2(\omega)$ telles que pour tout $T > 0$ suffisamment grand il existe un contrôle v dans $L^2(0, T)$ pour le système :

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$$(\mathcal{S}_c)\quad \begin{cases} \boldsymbol{y}'' + \mathbf{A}_c\boldsymbol{y} = \mathbf{0} & \text{dans } \omega \times (0,T),\\ y_1(0,t) = 0,\ y_1(1,t) = v(t) & \text{dans } (0,T),\\ \boldsymbol{y}(\xi,0) = \boldsymbol{y}^{\mathbf{0}},\quad \boldsymbol{y}'(\xi,0) = \boldsymbol{y}^{\mathbf{1}} & \text{dans } \omega, \end{cases} \tag{1}$$

i.e. des conditions initiales telles que l'unique solution de ce problème vérifie,

$$\boldsymbol{y}(\xi,T) = \mathbf{0},\ \boldsymbol{y}'(\xi,T) = \mathbf{0} \quad \text{dans } \omega.$$

L'opérateur auto-adjoint $\mathbf{A}_c$ est associé à la forme bilinéaire a_c définie sur $\mathbf{V} \times \mathbf{V}$ par,

$$a_c(\boldsymbol{y},\boldsymbol{v}) = \int_\omega \big(y_{1,1}(\xi) + c y_2(\xi)\big)\big(v_{1,1}(\xi) + c v_2(\xi)\big)\,\mathrm{d}\xi, \tag{2}$$

où $y_{i,1}(\xi) = \partial y_i(\xi)/\partial\xi$. L'étude est ramenée à l'analyse de la propriété d'observabilité pour le problème homogène adjoint :

$$\begin{cases} \boldsymbol{\phi} \in C(0,T;\mathbf{V}) \cap C^1(0,T;\mathbf{H}),\\ \boldsymbol{\phi}'' + \mathbf{A}_c\boldsymbol{\phi} = \mathbf{0} \quad \text{in } (0,T),\\ (\boldsymbol{\phi}(0), \boldsymbol{\phi}'(0)) = (\boldsymbol{\phi}^{\mathbf{0}}, \boldsymbol{\phi}^{\mathbf{1}}) \end{cases} \tag{3}$$

à données initiales dans $\boldsymbol{V} \times \boldsymbol{H}$. L'opérateur $\mathbf{A}_c$ a pour spectre $\{0, c^2, c^2 + k^2\pi^2\}$, $k > 0$. Il apparaît que la valeur propre 0 est de multiplicité infinie de sorte que 0 appartient au spectre essentiel $\sigma_{\mathrm{ess}}(\mathbf{A}_c)$ de l'opérateur, tandis que les valeurs c^2 et $c^2 + k^2\pi^2$ sont de multiplicité 1. On peut représenter la solution de (3) par le développement en série (8). L'utilisation d'un résultat d'Ingham permet alors d'obtenir l'observabilité de (3) pour toute donnée initiale dans $\widetilde{\mathbf{V}}_{\mathbf{0},\perp} \times \widetilde{\mathbf{H}}_{\mathbf{0}}^{\perp}$, où $\widetilde{\mathbf{H}}_{\mathbf{0}}^{\perp}$, définie par (11), est l'orthogonal du noyau de $\mathbf{A}_c$ et $\widetilde{\mathbf{V}}_{\mathbf{0},\perp}$ est l'orthogonal de ce noyau dans $\mathbf{V}$ (Proposition 2.1). Il est à remarquer que pour ces données intiales la solution $\boldsymbol{\phi}$ de (3) vérifie $\boldsymbol{\phi}(t) \in \widetilde{\mathbf{V}}_{\mathbf{0},\perp}$ pour tout $t > 0$. Par ailleurs, la présence de la valeur propre c^2 empêche l'observabilité uniforme lorsque c tend vers zéro illustrant le caractère singulier du mode $\mathbf{v}_0$ (Proposition 2.2). En utilisant la méthode HUM, [7], on obtient la contrôlabilité de (1) pour toute donnée initiale $(\boldsymbol{y}^{\mathbf{0}}, \boldsymbol{y}^{\mathbf{1}}) \in \mathbf{H}_c^{\perp} \times \widetilde{\mathbf{V}}_0^{\perp}$ (voir Théorème 3.1) où $\widetilde{\mathbf{V}}_0^{\perp}$ est la fermeture de $\widetilde{\mathbf{V}}_{\mathbf{0},\perp}$ dans $\boldsymbol{V}'$. On remarquera que $\mathbf{H}_c^{\perp}$, resp.$\widetilde{\mathbf{V}}_0^{\perp}$ est le sous-espace de $\mathbf{H}$, resp.$\mathbf{V}'$, engendré par les vecteurs propres associés aux valeurs propres de $\mathbf{A}_c$ satisfaisants la *condition de séparation* (Σ), voir (19). On sait, [4], que la calotte membranaire sphérique n'est pas, en général, exactement contrôlable. Dans ce cas aussi, avec une condition de séparation du même type (Σ), on peut identifier les données initiales exactement contrôlables, [1].

1. Statement of the problem

In [4] it has been considered the question of the exact controllability for a system of the form:

$$\begin{cases} \mathbf{u}'' + \mathbf{A}\mathbf{u} = 0, & \text{in} \quad \omega \times (0,T),\\ \mathbf{u}(0) = \mathbf{u}^0, & \mathbf{u}'(0) = \mathbf{u}^1,\\ \mathbf{B}\mathbf{u} = \mathbf{v} & \text{on } \partial\omega \times (0,T), \end{cases} \tag{4}$$

when $\mathbf{A}$ is a Douglis–Nirenberg elliptic system of mixed order, $\mathbf{B}$ is a system of normal boundary conditions; the realization of $\mathbf{A}$ associated to the boundary conditions $\mathbf{B}$ in the Hilbert space $\mathbf{H}$ is defined by a sesquilinear selfadjoint form $a(\mathbf{u},\mathbf{v})$ on $\mathbf{V} \times \mathbf{V}$. Exact (or null) controllability means that at a time T large enough the solution verifies:

$$\mathbf{u}(T) = \mathbf{0}, \qquad \mathbf{u}'(T) = \mathbf{0}. \tag{5}$$

It has been proved that when the operator $\mathbf{A}$ has a block of zero order, and hence $\mathbf{A}_{\mathbf{B}}$ has an essential spectrum (see [5]), there exist some initial data $(\mathbf{u}^0, \mathbf{u}^1)$ such that the system (4) is not exactly controllable. This is the situation, in particular, of membrane shells of arbitrary shapes. In this context we are interested in the following question:

(EC) *Characterize the space of initial data* $(\mathbf{u}^0, \mathbf{u}^1)$ *such that the system* (4) *is exactly controllable*

As a preliminary step in this Note we consider a particular 1D example. Let be $c \in \mathbb{R}^+$, $\omega = (0,1)$ and $\boldsymbol{y}(\xi) = (y_1(\xi), y_2(\xi))$ for $\xi \in \omega$. Let us consider in $\boldsymbol{H} = L^2(\omega) \times L^2(\omega)$ the operator $\mathbf{A}_c$ associated to the bilinear form $a_c(\boldsymbol{y},\boldsymbol{v})$ defined on $\mathbf{V} \times \mathbf{V}$ by (2) (where $\mathbf{V}$ is a suitable subspace of $\mathbf{H}$). One can take for simplicity $\boldsymbol{V} = H_0^1(\omega) \times$

$L^2(\omega)$. For $c = 0$, the bilinear form is simply $a_0(\boldsymbol{y}, \boldsymbol{v}) = \int_\omega y_{1,1} v_{1,1}\, d\xi$. In this Note we study the null-controllability of (1). Let us remark that the operator $\mathbf{A}_c$ is of Douglis–Nirenberg type with a zero order block and hence the results of [4] apply. We characterize *the class of initial data* such that the system $(\mathcal{S}_c)$ is exactly controllable; we also discuss the behavior of the control as the parameter c goes to zero.

2. Spectral property and observability

2.1. Spectral property and decomposition

A simple computation gives: $\sigma(\mathbf{A}_c) = \{0, \lambda_0 = c^2, \lambda_k = c^2 + k^2\pi^2, k = 1, 2, \ldots\}$. Since

$$\operatorname{Ker}\mathbf{A}_c = \left\{\mathbf{v}_\zeta = (\zeta, -c^{-1}\zeta_{,1}) \in \boldsymbol{H},\ \zeta \in H_0^1(\omega)\right\}, \tag{6}$$

and the eigenfunctions associated with $\lambda_0 = c^2$ and $\lambda_k = c^2 + k^2\pi^2$ are respectively:

$$\mathbf{v}_0 = (0, 1), \qquad \mathbf{v}_k = \frac{\sqrt{2}k\pi}{\sqrt{\lambda_k}}\left(\sin(k\pi\xi), \frac{c}{k\pi}\cos(k\pi\xi)\right), \tag{7}$$

one deduces that $\sigma_{\mathrm{ess}}(\mathbf{A}_c) = \{0\}$. Let us remark that an orthonormal (in $\mathbf{H}$) basis of $\operatorname{Ker}\mathbf{A}_c$ is:

$$\mathbf{w}_k = \frac{\sqrt{2}c}{\sqrt{\lambda_k}}\left(\sin(k\pi\xi), \frac{-k\pi}{c}\cos(k\pi\xi)\right), \quad k = 1, 2, \ldots$$

and that $\{\mathbf{w}_k, \mathbf{v}_0, \mathbf{v}_k\}$ is an orthonormal basis in $\mathbf{H}$.

The weak solution of the problem (3) can be expanded as follows, setting $\mu_0 = c$ and $\mu_k = \sqrt{\lambda_k}$:

$$\boldsymbol{\phi}(t) = \sum_{k=1}^{\infty}(a_k + b_k t)\mathbf{w}_k + \big(A_0\cos(\mu_0 t) + B_0\sin(\mu_0 t)\big)\mathbf{v}_0 + \sum_{k=1}^{\infty}\big(A_k\cos(\mu_k t) + B_k\sin(\mu_k t)\big)\mathbf{v}_k, \tag{8}$$

where the coefficients $a_k, b_k, A_0, B_0, A_k, B_k$ are determined from the expansion of the initial data:

$$\boldsymbol{\phi}^{\mathbf{0}} = \sum_{k=1}^{\infty} a_k\mathbf{w}_k + A_0\mathbf{v}_0 + \sum_{k=1}^{\infty} A_k\mathbf{v}_k, \quad \text{and} \quad \boldsymbol{\phi}^{\mathbf{1}} = \sum_{k=1}^{\infty} b_k\mathbf{w}_k + \mu_0 B_0\mathbf{v}_0 + \sum_{k=1}^{\infty}\mu_k B_k\mathbf{v}_k. \tag{9}$$

The assumption $(\boldsymbol{\phi}^{\mathbf{0}}, \boldsymbol{\phi}^{\mathbf{1}}) \in \mathbf{V} \times \mathbf{H}$ implies that

$$\sum_{k=1}^{\infty}\big(a_k^2 + A_k^2\big) < \infty, \quad \sum_{k=1}^{\infty}(ca_k + k\pi A_k)^2 < \infty, \quad \sum_{k=1}^{\infty}\big(b_k^2 + \lambda_k B_k^2\big) < \infty. \tag{10}$$

Observe that if $\boldsymbol{\phi}^{\mathbf{0}}, \boldsymbol{\phi}^{\mathbf{1}} \in \operatorname{Ker}\mathbf{A}_c$ then $\boldsymbol{\phi}(t) \in \operatorname{Ker}\mathbf{A}_c$ for all t. Similarly, if $\boldsymbol{\phi}^{\mathbf{0}} \in \operatorname{Ker}\mathbf{A}_c$ and $\boldsymbol{\phi}^{\mathbf{1}} = (0, 0)$, then $\boldsymbol{\phi}(t) = \boldsymbol{\phi}^{\mathbf{0}}$ for all $t > 0$.

The orthogonal of the subspace $\operatorname{Ker}\mathbf{A}_c$ in $\mathbf{H}$ is the following:

$$\mathbf{H}_c^{\perp} = \left\{\mathbf{v} = \left(c^{-1}\psi_{,1}, -\psi\right) \in \mathbf{H},\ \psi \in H^1(\omega)\right\}. \tag{11}$$

The subspace $\mathbf{H}_c^{\perp}$ is generated by $\mathbf{v}_0$ and $\mathbf{v}_k$, $k \geqslant 1$. We denote with $\mathbf{V}_{c,\perp}$ the orthogonal in $\mathbf{V}$ of $\operatorname{Ker}\mathbf{A}_c$; it is also generated, in $\mathbf{V}$, by $\mathbf{v}_0$ and $\mathbf{v}_k$, $k \geqslant 1$.

Let us remark that the energy,

$$\begin{aligned} E(t, \boldsymbol{\phi}) &= \frac{1}{2}\left\{\left\|\boldsymbol{\phi}'(t)\right\|_{\mathbf{H}}^2 + \int_\omega\big(\phi_{1,1}(\xi) + c\phi_2(\xi)\big)^2\,d\xi\right\} \\ &= E(0, \boldsymbol{\phi}) = \frac{1}{2}\left\{\left\|\boldsymbol{\phi}^1\right\|_{\mathbf{H}}^2 + \int_\omega\big(\phi_{1,1}^0(\xi) + c\phi_2^0(\xi)\big)^2\,d\xi\right\} \quad \forall t \in (0, T), \end{aligned} \tag{12}$$

defines a norm over $\mathbf{V}_{c,\perp} \times \mathbf{H}_c^{\perp}$. Moreover if $(\boldsymbol{\phi}^{\mathbf{0}}, \boldsymbol{\phi}^{\mathbf{1}}) \in \mathbf{V}_{c,\perp} \times \mathbf{H}_c^{\perp}$ then $\boldsymbol{\phi}(t) \in \mathbf{V}_{c,\perp}$ for all t.

2.2. Observability inequality

Let us recall that $\mathbf{B}$ is a normal system of boundary conditions when there exists a complementary system $\mathbf{C}$ such that $\{\mathbf{B}, \mathbf{C}\}$ are the reduced Cauchy data of $\mathbf{A}$; in our particular situation $\mathbf{C}\boldsymbol{\phi} = \phi_{1,1} + c\phi_2$. Since the control is only active at $\xi = 1$, the control property of the system (1) is related, see [4], to the existence of two positive constants C_1 and C_2 such that, for all $(\boldsymbol{\phi}^{\mathbf{0}}, \boldsymbol{\phi}^{\mathbf{1}}) \in \boldsymbol{V} \times \boldsymbol{H}$ and $T > 0$ large enough the solution of (3) satisfies:

$$C_1 \big\| (\boldsymbol{\phi}^{\mathbf{0}}, \boldsymbol{\phi}^{\mathbf{1}}) \big\|^2_{\boldsymbol{V} \times \boldsymbol{H}} \leqslant \int_0^T \big(\mathbf{C}\boldsymbol{\phi}(1,t)\big)^2 \,\mathrm{d}t \leqslant C_2 \big\| (\boldsymbol{\phi}^{\mathbf{0}}, \boldsymbol{\phi}^{\mathbf{1}}) \big\|^2_{\boldsymbol{V} \times \boldsymbol{H}}. \tag{13}$$

Since $0 \in \sigma(\mathbf{A}_c)$, the left inequality (called the observability inequality) cannot hold for all $(\boldsymbol{\phi}^{\mathbf{0}}, \boldsymbol{\phi}^{\mathbf{1}}) \in \boldsymbol{V} \times \boldsymbol{H}$. It suffices to take $\boldsymbol{\phi}^{\mathbf{0}}, \boldsymbol{\phi}^{\mathbf{1}} \in \operatorname{Ker} \mathbf{A}_c$ so that $\boldsymbol{\phi}(t) \in \operatorname{Ker} \mathbf{A}_c$ for all t and $\mathbf{C}\boldsymbol{\phi}(1,t) = 0$.

Proposition 2.1. *Let $c > 0$ and $\gamma^\star(c) = \min(2c, \sqrt{c^2 + \pi^2} - c)$. For all time $T > T^\star(c) \equiv 2\pi/\gamma^\star(c)$, there exist two strictly positive constants $C_1(c)$ and $C_2(c)$ such that (13) holds for all $(\boldsymbol{\phi}^{\mathbf{0}}, \boldsymbol{\phi}^{\mathbf{1}}) \in \widetilde{\mathbf{V}}_{\mathbf{0},\perp} \times \widetilde{\mathbf{H}}_{\mathbf{0}}^{\perp}$.*

Proof of Proposition 2.1. When $(\boldsymbol{\phi}^{\mathbf{0}}, \boldsymbol{\phi}^{\mathbf{1}}) \in \widetilde{\mathbf{V}}_{\mathbf{0},\perp} \times \widetilde{\mathbf{H}}_{\mathbf{0}}^{\perp}$ (13) is equivalent to,

$$C_1(c) E(0, \boldsymbol{\phi}) \leqslant \int_0^T \big(\phi_{1,1}(1,t) + c\phi_2(1,t)\big)^2 \,\mathrm{d}t \leqslant C_2(c) E(0, \boldsymbol{\phi}). \tag{14}$$

As usual in this context, these inequalities may be obtained from a direct application of a Ingham's theorem on non-harmonic series (see e.g. [6, p. 59]). Indeed, with an easy computation one finds:

$$E(0, \boldsymbol{\phi}) = \frac{c^2}{2}\big(A_0^2 + B_0^2\big) + \frac{1}{2}\sum_{k=1}^{\infty} \lambda_k \big(A_k^2 + B_k^2\big). \tag{15}$$

On the other hand, one has:

$$\begin{aligned} \phi_{1,1}(1,t) + c\phi_2(1,t) &= c\big(A_0 \cos(\mu_0 t) + B_0 \sin(\mu_0 t)\big) + \sqrt{2}\sum_{k=1}^{\infty} (-1)^k \mu_k^2 \big(A_k \cos(\mu_k t) + B_k \sin(\mu_k t)\big) \\ &= \frac{1}{\sqrt{2}} \sum_{k=1}^{\infty} (-1)^k \mu_k^2 (A_k + \mathrm{i}B_k)\,\mathrm{e}^{-\mathrm{i}\mu_k t} + \frac{c}{2}(A_0 + \mathrm{i}B_0)\,\mathrm{e}^{-\mathrm{i}\mu_0 t} \\ &\quad + \frac{c}{2}(A_0 - \mathrm{i}B_0)\,\mathrm{e}^{\mathrm{i}\mu_0 t} + \frac{1}{\sqrt{2}} \sum_{k=1}^{\infty} (-1)^k \mu_k^2 (A_k - \mathrm{i}B_k)\,\mathrm{e}^{\mathrm{i}\mu_k t}. \end{aligned} \tag{16}$$

We then apply the previously quoted Ingham's theorem with $I = (0, T)$ and the sequence,

$$w = (\ldots, -\mu_2, -\mu_1, -\mu_0, \mu_0, \mu_1, \mu_2, \ldots),$$

to obtain that the existence of two positives constants $C_1(c)$ and $C_2(c)$ such that (14) holds for all $c > 0$, under the condition $T > 2\pi/\gamma$, with $\gamma = \min(\mu_0 - (-\mu_0), \inf_{k \in \mathbb{N}} |\mu_k - \mu_{k-1}|)$. From the concavity of the square root function, we deduce that $|\mu_1 - \mu_0| \leqslant |\mu_{k+1} - \mu_k|$ for all $k \geqslant 0$ and then that $\gamma = \gamma^\star(c)$. □

The lower bound value $T^\star$ of observability time is reached when $c = \pi^2/8$ for which $T^\star = \sqrt{8}$. We observe that the time of controllability $T^\star$ blows up as c goes to zero. This is due to the eigenvalue λ_0 which vanishes with c. Precisely, if the initial condition is $(\boldsymbol{\phi}^{\mathbf{0}}, \boldsymbol{\phi}^{\mathbf{1}}) = (\mathbf{v}_0, \mu_0 \mathbf{v}_0)$ so that $\phi_{1,1}(1,t) + c\phi_2(1,t) = c(\cos(\mu_0 t) + \sin(\mu_0 t))$ then we obtain explicitly that the constant $C_1(c)$ goes to zero as c goes to zero unless $T = \mathrm{O}(c^{-1})$. Consequently, the observability inequality is not uniform with respect to c. If we denote with $\mathbf{H}_K$, resp. $\mathbf{V}_K$, the closed subspace of $\mathbf{H}$, resp. $\mathbf{V}$, generated by $\mathbf{v}_k$ for all $k \geqslant 1$, we have the following result:

Proposition 2.2 *(Uniform observability w.r.t. c). Let $c > 0$ and $\gamma^{\star\star}(c) = \sqrt{c^2 + 4\pi^2} - \sqrt{c^2 + \pi^2}$. For all $T > T^{\star\star}(c) \equiv 2\pi/\gamma^{\star\star}(c)$, there exist two positive constants C_1 and C_2 independent of c such that* (13) *holds for all $(\boldsymbol{\phi}^{\mathbf{0}}, \boldsymbol{\phi}^{\mathbf{1}}) \in \mathbf{V}_K \times \mathbf{H}_K$.*

The lower bound $T^{\star\star}$ is now a monotonous increasing function of c such that $\lim_{c\to 0} T^{\star\star}(c) = 2$, lower bound for the wave equation controlled at one extremity. We also remark that $2 < T^{\star\star}(c) < T^{\star}(c)$ for all $c > 0$.

3. Exact controllability

We now apply the Hilbert Uniqueness Method of J.-L. Lions [7]. Let be $\boldsymbol{V'} = H^{-1}(\omega) \times L^2(\omega)$ so that one has the usual situation $\boldsymbol{V} \subset \boldsymbol{H} \approx (\boldsymbol{H})' \subset \boldsymbol{V}'$ with $\mathbf{H}$ as pivot space. Let us assume that $(\boldsymbol{y}^{\mathbf{0}}, \boldsymbol{y}^{\mathbf{1}}) \in \boldsymbol{H} \times \boldsymbol{V'}$. Formal integrations by part provide that v is a control for the system (1) if and only if

$$\int_0^T (\phi_{1,1} + c\phi_2)(1,t)v(t)\,\mathrm{d}t = \langle \boldsymbol{\phi}^{\mathbf{0}}, \boldsymbol{y}^{\mathbf{1}}\rangle_{V,V'} - (\boldsymbol{\phi}^{\mathbf{1}}, \boldsymbol{y}^{\mathbf{0}})_{\mathbf{H}} \tag{17}$$

for every $\boldsymbol{\phi}$ solution of (3). We then introduce the continuous and convex functional $\mathcal{J}_c : \boldsymbol{V} \times \boldsymbol{H} \to \mathbb{R}$ defined by:

$$\mathcal{J}_c(\boldsymbol{\phi}^{\mathbf{0}}, \boldsymbol{\phi}^{\mathbf{1}}) = \frac{1}{2}\int_0^T (\phi_{1,1} + c\phi_2)^2(1,t)\,\mathrm{d}t - \langle \boldsymbol{\phi}^{\mathbf{0}}, \boldsymbol{y}^{\mathbf{1}}\rangle_{V,V'} + (\boldsymbol{\phi}^{\mathbf{1}}, \boldsymbol{y}^{\mathbf{0}})_{\mathbf{H}}. \tag{18}$$

If $\mathcal{J}_c$ is coercive, then $\mathcal{J}_c$ admits a unique minimum and the HUM control of minimal $L^2(0,T)$-norm is given by $v_c = (\phi_{1,1} + c\phi_3)(1,\cdot)$. Let us remark at first that $\mathcal{J}_c$ is only coercive on $\widetilde{\mathbf{V}}_{\mathbf{0},\perp} \times \widetilde{\mathbf{H}}_{\mathbf{0}}^{\perp}$ (provided T be large enough). Furthermore, if $(\boldsymbol{y}^{\mathbf{0}}, \boldsymbol{y}^{\mathbf{1}})$ belongs to $\operatorname{Ker}\mathbf{A}_c$, then $\langle \boldsymbol{\phi}^{\mathbf{0}}, \boldsymbol{y}^{\mathbf{1}}\rangle_{V,V'} - (\boldsymbol{\phi}^{\mathbf{1}}, \boldsymbol{y}^{\mathbf{0}})_{\mathbf{H}} = 0$ for all $(\boldsymbol{\phi}^{\mathbf{0}}, \boldsymbol{\phi}^{\mathbf{1}}) \in \widetilde{\mathbf{V}}_{\mathbf{0}\perp} \times \widetilde{\mathbf{H}}_{\mathbf{0}}^{\perp}$ and from (17), the control is zero; in this case, the solution $\boldsymbol{y}$ remains in $\operatorname{Ker}\mathbf{A}_c$ for all $t > 0$ but is not controlled! Hence we take $\boldsymbol{y}^{\mathbf{0}} \in \mathbf{H}_c^{\perp}$ and $\boldsymbol{y}^{\mathbf{1}} \in (\widetilde{\mathbf{V}}_0)^{\perp}$ where $(\widetilde{\mathbf{V}}_0)^{\perp} \subset \boldsymbol{V}'$ is the orthogonal of $\widetilde{\mathbf{V}}_0$ in the duality $\langle\cdot,\cdot\rangle_{V,V'}$ and we have:

Theorem 3.1. *Let $c > 0$. For any $T > T^{\star}(c)$ and any initial data $(\boldsymbol{y}^{\mathbf{0}}, \boldsymbol{y}^{\mathbf{1}}) \in \mathbf{H}_c^{\perp} \times \widetilde{\mathbf{V}}_0^{\perp}$, there exists a control function $v \in L^2(0,T)$ which drives to rest at time T the solution $\boldsymbol{y}$ of* (1) *associated with $(\boldsymbol{y}^{\mathbf{0}}, \boldsymbol{y}^{\mathbf{1}})$. Moreover, the control of minimal L^2-norm is given by $v(t) = (\phi_{1,1} + c\phi_2)(1,t)$, where $\boldsymbol{\phi}$ is the solution of* (3) *with $(\boldsymbol{\phi}^{\mathbf{0}}, \boldsymbol{\phi}^{\mathbf{1}})$ minimum of $\mathcal{J}_c$ defined by* (18) *over $\widetilde{\mathbf{V}}_{\mathbf{0},\perp} \times \widetilde{\mathbf{H}}_{\mathbf{0}}^{\perp}$.*

Let us explicitly remark that $\widetilde{\mathbf{V}}_0^{\perp}$ is the closure of $\widetilde{\mathbf{V}}_{\mathbf{0},\perp}$ in $\boldsymbol{V'}$. From Proposition 2.2, one has the following, where $\widehat{\mathbf{V}}_K$ is the closure of $\mathbf{V}_K$ in $\boldsymbol{V'}$:

Theorem 3.2 *(Uniform controllability w.r.t. c). Let $c > 0$. For any $T > T^{\star\star}(c)$ and any initial data $(\boldsymbol{y}^{\mathbf{0}}, \boldsymbol{y}^{\mathbf{1}}) \in \mathbf{H}_K \times \widehat{\mathbf{V}}_K$, there exists a control function $v \in L^2(0,T)$ which drives to rest at time T the solution $\boldsymbol{y}_c$ of* (1) *associated with $(\boldsymbol{y}^{\mathbf{0}}, \boldsymbol{y}^{\mathbf{1}})$. Moreover, the control of minimal L^2-norm is given by $v_c(t) = (\phi(c)_{1,1} + c\phi(c)_2)(1,t)$ where $\boldsymbol{\phi}(c)$ is solution of* (3) *with $(\boldsymbol{\phi}^{\mathbf{0}}, \boldsymbol{\phi}^{\mathbf{1}})$ minimum of $\mathcal{J}_c$ defined by* (18) *over $\mathbf{V}_{K,} \times \mathbf{H}_K$. At last, this control converges weakly in $L^2(0,T)$ as c goes to zero toward the control of minimal L^2-norm which drives to rest the solution of the wave equation associated with (y_1^0, y_1^1).*

4. Concluding remarks

1. System (S_c) is, for $c > 0$, null controllable by a Dirichlet force only when the initial data $(\boldsymbol{y}^{\mathbf{0}}, \boldsymbol{y}^{\mathbf{1}})$ are in $\mathbf{H}_c^{\perp} \times \widetilde{\mathbf{V}}_0^{\perp}$. Let us point out that the subspaces $\mathbf{H}_c^{\perp}$ of $\mathbf{H}$ and $\widetilde{\mathbf{V}}_0^{\perp}$ of $\boldsymbol{V'}$ are spanned by the eigenfunctions associated to the sequence of eigenvalues λ_k whose elements have the following *separation property*:

$$(\Sigma) \quad \text{there exists } \lambda^* > 0 \text{ such that } \operatorname{dist}(\{\lambda_k\}_{k\geqslant 0}, \sigma_{\mathrm{ess}}(\mathbf{A}_c)) = \lambda^* \text{ and } \lim_{k\to\infty} \lambda_k = +\infty. \tag{19}$$

2. The analysis also reveals that the time of controllability increases when c goes to zero and that there exists a non-controllable mode for $c = 0$. Also in this case the subspace of controllable initial data is spanned by the eigenfunctions associated to the eigenvalues satisfying a condition of type (Σ), [1].

3. Analogous conclusions are true for a Neumann control $\mathbf{Cy} = y_{1,1} + cy_2$ at one extremity, [1].

4. In [2] and [3] it has been proved that the exact controllability problem for spherical membrane fails when the vibrations are due to external actions of wavelengths connected to the eigenvalues with finite accumulation point (this point is exactly the essential spectrum of the operator). One can prove that the subspace of controllable initial data also in this case is spanned by the eigenfunctions associated with the sequence of eigenvalues satisfying the condition (Σ), [1].

5. The characterization of the initial conditions that are controllable for the membrane shell that have essential spectrum, see [8], is open.

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