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No characterization of generators in ℓ^p $(1 < p < 2)$ by zero set of Fourier transform

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Abstract

Given $1 < p < 2$ we construct two continuous functions f and g on the circle, with the following properties:
(i) They have the same set of zeros;
(ii) The Fourier transforms $\hat{f}$ and $\hat{g}$ both belong to $\ell^p(\mathbb{Z})$;
(iii) The translates of $\hat{g}$ span the whole ℓ^p, but those of $\hat{f}$ do not.
A similar result is true for $L^p(\mathbb{R})$. This should be contrasted with the Wiener theorems related to $p = 1, 2$. ***To cite this article: N. Lev, A. Olevskii, C. R. Acad. Sci. Paris, Ser. I 346 (2008).***
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Résumé

Les générateurs de ℓ^p $(1 < p < 2)$ ne peuvent pas être caractérisés par une propriété de l'ensemble des zéros de leur transformées de Fourier. Étant donné $1 < p < 2$ nous construisons deux fonctions continues sur le cercle, f et g, telles que :
(i) Elles ont le même ensemble de zéros ;
(ii) Leurs transformées de Fourier appartiennent à $\ell^p(\mathbb{Z})$;
(iii) Les translatées de la transformée de Fourier de g engendrent ℓ^p, mais non celles de la transformées de Fourier de f.
Un résultat analogue est valable pour $L^p(\mathbb{R})$. Cela contraste avec les cas $p = 1$ ou 2, élucidés par Wiener. ***Pour citer cet article : N. Lev, A. Olevskii, C. R. Acad. Sci. Paris, Ser. I 346 (2008).***
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1. Introduction and results

1.1. A function $F : \mathbb{Z} \to \mathbb{C}$ is called a cyclic vector, or a generator, in the space $\ell^p(\mathbb{Z})$ if the linear span of its translates is dense in the space. For $p = 1$ and 2, Wiener characterized the generators by the zero set Z_f of the Fourier transform

$$f(t) := \sum_{n \in \mathbb{Z}} F(n) \mathrm{e}^{\mathrm{int}}, \quad t \in \mathbb{T} := \mathbb{R}/2\pi\mathbb{Z},$$

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as follows:

F is a generator in ℓ^2 if and only if $f(t)$ is non-zero a.e.

F is a generator in ℓ^1 if and only if $f(t)$ has no zeros.

The same characterization holds for $L^2(\mathbb{R})$ and $L^1(\mathbb{R})$, see [7].

"Interpolating" between $p = 1$ and 2 one may expect that the generators in ℓ^p (or L^p), $1 < p < 2$, could be characterized by the condition that the zero set of the Fourier transform is "small" in a certain sense. In this context various metrical, arithmetical and other properties of the zero set for generators and non-generators have been studied by Beurling [1], Pollard [6], Herz [2], Newman [5] and other authors. However, none of these results provide a complete characterization of generators.

We will prove that such a characterization is impossible in principle. The following theorem is true:

Theorem 1. *Given $1 < p < 2$ one can find two continuous functions f and g on the circle $\mathbb{T}$, with the following properties*:

(i) $\{t\colon\ f(t) = 0\} = \{t\colon\ g(t) = 0\}$,
(ii) *$F := \hat{f}$ and $G := \hat{g}$ are both in $\ell^p(\mathbb{Z})$,*
(iii) *G is a generator in ℓ^p, but F is not.*

Remarks.

1. The role of the continuity condition is to make certain the concept of the "zero set".
2. In fact the function f in Theorem 1 can be chosen smooth. However, f and g cannot both be smooth.

The L^p version is also true:

Theorem 1′. *Given $1 < p < 2$ one can find two functions F and G in $L^p(\mathbb{R})$ with the following properties. The Fourier transforms $\hat{F}(t)$, $\hat{G}(t)$ are continuous functions on $\hat{\mathbb{R}}$; they have the same zero set; the set of translates $\{G(x - u)\}$, $u \in \mathbb{R}$, spans the whole space, but $\{F(x - u)\}$ does not.*

1.2. Denote by $A_r(\mathbb{T})$ $(1 \leqslant r < \infty)$ the Banach space of functions or distributions on the circle with Fourier coefficients in $\ell^r(\mathbb{Z})$, endowed with the norm $\|f\|_{A_r} := \|\hat{f}\|_{\ell^r}$. Our main result can be formulated as follows:

Theorem 2. *For any $1 < p < 2$ one can construct a compact $E \subset \mathbb{T}$, and a function $g \in C(\mathbb{T}) \cap A_p(\mathbb{T})$, such that*:

(a) $Z_g := \{t\colon\ g(t) = 0\} = E$;
(b) *The set $\{P(t)g(t)\}$, where P goes through all trigonometric polynomials, is dense in A_p;*
(c) *There is a (non-zero) distribution S, supported by E, which belongs to A_q, $q = p/(p - 1)$.*

Clearly (b) means that $\hat{g}$ is a generator. On the other hand (c) is equivalent to the fact that no Fourier transform of a smooth function f vanishing on E, could be a generator. So Theorem 1 is a direct consequence of Theorem 2. Theorem 1′ also follows.

Theorem 2 strengthens our result from [4], where we constructed a compact E which supports a distribution belonging to A_q $(q > 2)$, but does not support such a measure. Clearly the compact E from Theorem 2 satisfies this property.

The proof of Theorem 2 sketched below is based on a modification and development of the approach used in [4].

2. Riesz-type products

2.1. As in [4] we consider finite Riesz products, but instead of the cosine function we now use a certain trigonometric polynomial φ, taken from the following:

Lemma 1. *Given $0 < \eta < 1$ there is a real trigonometric polynomial $\varphi = \varphi_\eta$ such that*

$$\hat{\varphi}(0) = 0, \quad \|\varphi\|_\infty = 1, \quad \|\varphi\|_{L^2} > \frac{9}{10}, \quad \|\varphi\|_{A_p} \leqslant C\eta^{-1}, \quad \|\varphi\|_{A_q} \leqslant C\eta$$

(*here C is an absolute constant*).

2.2. For every $s \in I := (\frac{8}{10}, \frac{9}{10})$ define

$$\lambda_s(t) = \prod_{j=1}^{N} \bigl(1 + s\varphi(\nu^j t)\bigr).$$

After opening the brackets one gets

$$\lambda_s(t) = 1 + \sum \left\{ s^l \prod_{k_j \neq 0} \hat{\varphi}(k_j) \right\} \mathrm{e}^{\mathrm{i}(k_1\nu + k_2\nu^2 + \cdots + k_N\nu^N)t},$$

where the sum is taken over all non-zero vectors $k = (k_1, k_2, \ldots, k_N)$ such that $|k_j| \leqslant \deg\varphi$, $l = l(k) > 0$, and all the frequencies are distinct provided that $\nu > 2\deg\varphi$. We then integrate λ_s against a measure ρ, supported by I, which has the zero moment equal to 1 and all other moments less than δ by modulus (see [3], p. 214). The A_q norm of the resulting function,

$$\lambda(t) = \int \lambda_s(t)\,\mathrm{d}\rho(s),$$

could be estimated as

$$\sum_{n \neq 0} \bigl|\hat{\lambda}(n)\bigr|^q < \delta \sum \prod_{k_j \neq 0} \bigl|\hat{\varphi}(k_j)\bigr|^q < \delta\bigl(1 + \|\varphi\|_{A_q}^q\bigr)^N.$$

3. Concentration

3.1. Define a trigonometric polynomial

$$X(t) = \frac{1}{N} \sum_{j=1}^{N} \varphi(\nu^j t).$$

One can see that for a sufficiently large ν, depending on N and η, the members of this polynomial are "almost" stochastically independent on $\mathbb{T}$ with respect to the probability measure

$$\mathrm{d}\mu_s(t) := \lambda_s(t)\,\mathrm{d}t/2\pi.$$

The classical Bernstein inequality thus implies the exponential estimate

$$\mathrm{prob}\bigl(\bigl|X(t) - \mathcal{E}X\bigr| > \alpha\bigr) < 3\exp\left(-\frac{1}{8}\alpha^2 N\right), \quad \nu > \nu(N, \eta),$$

which holds for every $s \in I$. Using the estimates

$$\mathcal{E}X > \frac{5}{8}, \qquad \lambda_s(t) \leqslant \exp\bigl(sNX(t)\bigr)$$

one can prove:

Lemma 2. *Given $\delta > 0$ there is $N(\delta)$ such that, for every $N \geqslant N(\delta)$ and $\nu > \nu(N, \eta)$,*

$$\int\limits_{\{t:\, X(t) < \frac{1}{40}\}} \lambda_s^2(t) \frac{\mathrm{d}t}{2\pi} < \delta \quad (s \in I).$$

3.2. Now we can formulate the main lemma:

Lemma 3. *Given $\varepsilon > 0$ there exist a compact K (a finite union of segments) on the circle, a smooth function F and a real trigonometric polynomial X such that:*

(i) *F is supported by K, $\|1 - F\|_{A_q} < \varepsilon$,*
(ii) *$\|X\|_\infty \leqslant 1$, $\|X\|_{A_p} < \varepsilon$, $X(t) > \frac{1}{50}$ on K.*

The proof follows the same line as the proof of Lemma 3.2 in [4], but here it takes advantage of the better estimates for λ and X.

4. Approximations

4.1. Lemma 3 allows us to produce successive approximations to the function g and the distribution S of Theorem 2. It is now possible to define inductively a sequence of smooth functions $\{f_n\}$, supported by compacts K_n (each next compact is embedded into the previous one), such that

$$f_0 = 1, \qquad \|f_n - f_{n-1}\|_{A_q} < 2^{-n-1}, \tag{1}$$

and simultaneously a sequence of non-zero trigonometric polynomials $\{g_n\}$ satisfying

$$\|g_n - g_{n-1}\|_\infty < c^{n-1}, \tag{2}$$

$$\sup_{t \in K_n} |g_n(t)| < c^n, \tag{3}$$

where the constant $c = \frac{99}{100}$.

Let us describe the n-th step of the induction. First we choose a trigonometric polynomial h such that

$$\sup_{t \in K_n} |g_n(t) - h(t)| < (1 - c)c^n, \qquad \|h\|_\infty < c^n.$$

Then taking a sufficiently small ε we use Lemma 3 to choose K, F and X, and set

$$f_{n+1} := f_n \cdot F, \qquad K_{n+1} := K_n \cap K, \qquad g_{n+1} := g_n - h \cdot X.$$

4.2. The estimate (1) implies that f_n will converge in A_q to a distribution S supported by $\bigcap_{n=1}^\infty K_n$. On the other hand g_n converges uniformly to some $g \in C(\mathbb{T})$, due to (2), and g vanishes on the support of S due to (3). Taking $E := Z_g$ one gets (a) and (c) in Theorem 2. Finally, notice that by taking ε small enough on each step of the induction, we may have

$$\|g_{n+1} - g_n\|_{A_p} \text{ decrease arbitrarily fast.} \tag{4}$$

Since g_n is a non-zero trigonometric polynomial, there is a polynomial $P_n(t)$ such that $\|1 - P_n \cdot g_n\|_{A_p} < 1/n$, and (4) allows us to replace here g_n by g. This easily implies (b).

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