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Differential Geometry

Gagliardo–Nirenberg inequalities involving the gradient L^2-norm

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Abstract

We present a method giving the sharp constants and optimal functions of all the Gagliardo–Nirenberg inequalities involving the L^2-norm of the gradient. We show that the optimal functions can be explicitly derived from a specific non-linear ordinary differential equation which appears to be linear for a subclass of the Gagliardo–Nirenberg inequalities or when the space dimension reduces to 1. In these cases, we give the explicit expressions of the optimal functions, along with the sharp constants of the corresponding Gagliardo–Nirenberg inequalities. Our method extend to the L^p-Gagliardo–Nirenberg and L^p-Nash's inequalities, for all $p > 1$. ***To cite this article: M. Agueh, C. R. Acad. Sci. Paris, Ser. I 346 (2008).***

Résumé

Inégalites de Gagliardo–Nirenberg optimales. Nous présentons une méthode donnant les constantes et fonctions optimales de toutes les inégalités de Gagliardo–Nirenberg dépendant de la norme L^2 du gradient. Nous montrons que les fonctions optimales se calculent explicitement à partir d'une équation différentielle ordinaire nonlinéaire, qui devient linéaire pour une sous-classe de ces inégalités ou quand la dimension de l'espace est réduite a 1. Dans ces cas, nous obtenons explicitement les fonctions et constantes optimales des inégalités de Gagliardo–Nirenberg correspondantes. Notre méthode se généralise aux inégalités de Gagliardo–Nirenberg et de Nash dependant de la norme L^p du gradient, pour tout $p > 1$. ***Pour citer cet article : M. Agueh, C. R. Acad. Sci. Paris, Ser. I 346 (2008).***

Version française abrégée

Dans cette Note, nous nous intéressons aux constantes et fonctions optimales des inégalités de Gagliardo–Nirenberg, [7,9], qui sont des inégalités de la forme

$$\|u\|_{L^p(\mathbb{R}^n)} \leqslant K_{\text{opt}} \|\nabla u\|^{\theta}_{L^2(\mathbb{R}^n)} \|u\|^{1-\theta}_{L^q(\mathbb{R}^n)} \quad \forall u \in D^{1,q}(\mathbb{R}^n) \tag{1}$$

où $K_{\text{opt}} > 0$, $n > 2$ et $1 < q < p < 2^* := \frac{2n}{n-2}$, ou $n = 1, 2$ et $1 < q < p$, et $\theta = \frac{2n(p-q)}{p[2n-q(n-2)]}$. Ici, $D^{1,q}(\mathbb{R}^n) := \{u \in L^q(\mathbb{R}^n)\colon\ \nabla u \in L^2(\mathbb{R}^n)\}$. Récemment, ce probème a été étudié dans beaucoup d'articles [8,6,5,3], et des progrès

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significatifs ont été faits dans cette direction. Mais les résultats obtenus sont limités à une sous-classe de ces inégalités, notamment, celle où les fonctions optimales sont des puissances polynomiales ou rationelles. Quand $n \geqslant 2$, il s'agit présicément des cas $q = 1 + \frac{p}{2}$ et $q = 2(p-1)$, où les constantes et fonctions optimales sont récemment obtenues par Del Pino et Dolbeault [6]. Ici, nous considérons l'inégalité (1) en général, même si les conditions $q = 1 + \frac{p}{2}$ et $q = 2(p-1)$ ne sont pas satisfaites. En dimension $n = 1$, nous avons entièrement résolu ce problème dans [1], et de plus, nous y avons établi le lien entre ces inégalités et la théorie de Transport de masse. Dans cet article, nous généralisons la méthode de [1] en dimensions supérieures, $n \geqslant 2$. Dans l'espoir de rendre notre exposé simple et claire, nous allons nous restreindre aux inégalités de Gagliardo–Nirenberg qui sont fonction de la norme L^2 du gradient, c'est-à-dire (1), bien que notre méthode se généralise à toutes les inégalités de Gagliardo–Nirenberg qui dependent de la norme L^r du gradient, où $1 < r < n$ (voir [2]).

1. Introduction

The present Note deals with the Gagliardo–Nirenberg inequalities, [7,9], which are geometric inequalities of the form (1), where $K_{\text{opt}} > 0$, $n > 2$ and $1 < q < p < 2^* := \frac{2n}{n-2}$, or $n = 1, 2$ and $1 < q < p$, and $\theta = \frac{2n(p-q)}{p[2n-q(n-2)]}$. Here,

$$D^{1,q}(\mathbb{R}^n) := \left\{u \in L^q(\mathbb{R}^n)\colon \nabla u \in L^2(\mathbb{R}^n)\right\},$$

and we consider the L^2-norm of ∇u for simplicity, though in general, the Gagliardo–Nirenberg inequalities can be stated with the L^r-norm of ∇u, where $1 < r < n$. The problem of finding the sharp constants and optimal functions of these inequalities has attracted many researchers in the past few years, [8,6,5,3]. Though significant progress was made on this subject, the results obtained so far are restricted to a special subclass of these inequalities, namely, those for which the optimal functions involve only power laws. When $n \geqslant 2$, this is precisely the cases $q = 1 + \frac{p}{2}$ and $q = 2(p-1)$, where the sharp constants and optimal functions are recently obtained by Del-Pino and Dolbeault in [6]. Here, we address the issue of the sharp constants and optimal functions of the Gagliardo–Nirenberg inequality (1) in general, that is, even if the condition $q = 1 + \frac{p}{2}$ or $q = 2(p-1)$ is not satisfied. In the 1-dimensional setting, the sharp constants and optimal functions of inequality (1) are recently derived in general by the author in [1], and the link between the inequality and Mass transportation theory is discussed. The present paper extends to higher dimensions, $n \geqslant 2$, the ideas presented in [1]. For simplicity, we will restrict to the L^2-Gagliardo–Nirenberg inequalities, though our analysis does apply to all L^r-Gagliardo–Nirenberg inequalities for $1 < r < n$, [2]. Here is a brief sketch of our method; for more details, we refer to [2]. Gagliardo–Nirenberg inequality (1), in its sharp form, follows directly from the variational problem

$$\inf\left\{E(u) := \frac{1}{2}\int_{\mathbb{R}^n} |\nabla u|^2\,\mathrm{d}x + \frac{1}{q}\int_{\mathbb{R}^n} |u|^q\,\mathrm{d}x\colon\ u \in D^{1,q}(\mathbb{R}^n),\ \|u\|_{L^p(\mathbb{R}^n)} = 1\right\}, \tag{2}$$

as soon as one can determine, explicitly, a minimizer to this problem, (see Theorem 2.1). Although the existence of a minimizer to problem (2) is not hard to show (see Theorem 3.1), computing explicitly a minimizer appears very difficult, as it involves solving the non-linear PDE

$$-\Delta u + u^{q-1} - \lambda u^{p-1} = 0, \tag{3}$$

where $\lambda > 0$ denotes the Lagrange multiplier for the constraint $\|u\|_{L^p(\mathbb{R}^n)} = 1$. This is where lies the main difficulty of the problem. By a rearrangement argument, it can be shown that solving PDE (3) is equivalent to finding the unique non-negative decreasing solution of the ODE

$$v''(r) + (n-1)\frac{v'(r)}{r} - v^{q-1}(r) + v^{p-1}(r) = 0, \tag{4}$$

where v and u are related by $v(r) = \lambda^{\frac{1}{p-q}} u(\lambda^{\frac{q-2}{2(p-q)}} x)$, $r = |x|$ (see Theorem 3.1). Then, there exits a – change of – function $H : (0, v(0)) \to \mathbb{R}$, such that $H(v(r)) = \frac{r^2}{2}$. This change of function is suggested by the link between certain Gagliardo–Nirenberg inequalities and Mass transportation theory (see details in [2]). Using this change of function in (4), we show that H satisfies the non-linear ODE

$$2\left(\frac{t^q}{q} - \frac{t^p}{p}\right)H''(t) + (t^{q-1} - t^{p-1})H'(t) - 2(n-1)H''(t)\int_0^t \frac{\mathrm{d}s}{H'(s)} = n, \tag{5}$$

whose solution gives an explicit minimizer to (2) (see Theorem 3.2), and therefore determines the sharp constants and optimal functions of all the Gagliardo–Nirenberg inequalities (1). We observe that if $n = 1$, or if we assume that $H''(t)\int_0^t \frac{ds}{H'(s)}$ is constant when $n \geqslant 2$, then (5) reduces to a first order linear ODE in H', which can be solved explicitly for all values of p and q. Therefore, when $n = 1$, we obtain the sharp constants and optimal functions of all the Gagliardo–Nirenberg inequalities (1) for $1 < q < p$ (see Corollary 3.3). When $n \geqslant 2$, we show that the condition $H''(t)\int_0^t \frac{ds}{H'(s)} = \text{constant}$, leads to the subclass of the Gagliardo–Nirenberg inequalities where $q = 1 + \frac{p}{2}$ or $q = 2(p-1)$. In these cases, we recover previous results obtained in [6] (see Corollary 3.4). Our method shows that when $n \geqslant 2$, the sharp constants and optimal functions of the Gagliardo–Nirenberg inequalities (1) in the cases $q = 1 + \frac{p}{2}$ and $q = 2(p-1)$ follow from a linear first order ODE, while the remaining Gagliardo–Nirenberg inequalities require solving the non-linear ODE (5), which is certainly more involved. We point out that our analysis generalizes to all the L^r-Gagliardo–Nirenberg inequalities for $1 < r < n$, [2]. Finally, to see how these inequalities link to Mass transportation theory, we refer to [1,2]. Throughout the paper, $\|u\|_r$ denotes the L^r-norm of $u : \mathbb{R}^n \to \mathbb{R}$, χ_A stands for the characteristic function of $A \subset \mathbb{R}^n$, and $\operatorname{sign}(u)$ is the sign of u.

2. Sharp constants in Gagliardo–Nirenberg/Nash's inequalities

In this section, we derive the sharp constant of the Gagliardo–Nirenberg/Nash's inequality (1), assuming that the variational problem (2) has a minimizer. The existence of a minimizer to this problem will be discussed in the next section (see Theorem 3.1).

Theorem 2.1. *Let n, p, q be such that $1 \leqslant q < p < 2^* := \frac{2n}{n-2}$ if $n > 2$, and $1 \leqslant q < p$ if $n = 1, 2$. Assume that the variational problem* (2) *has a minimizer u_∞. Then the Gagliardo–Nirenberg/Nash's inequality* (1) *holds, with $\theta = \frac{2n(p-q)}{p[2n-q(n-2)]}$, and the sharp constant $K_{\text{opt}} > 0$ is explicitly given by*

$$K_{\text{opt}} = \left[\frac{K(n,p,q)}{E(u_\infty)}\right]^{\frac{2n+2p-nq}{p[2n-q(n-2)]}}, \tag{6}$$

where $K(n,p,q) = \frac{\alpha+\beta}{(q\alpha)^{\frac{\alpha}{\alpha+\beta}}(2\beta)^{\frac{\beta}{\alpha+\beta}}}$, $\alpha = 2n - p(n-2)$, $\beta = n(p-q)$.

Moreover, $u_{\sigma,\bar{x}}(x) = Cu_\infty(\sigma(x-\bar{x}))$ are extremals in (1), *for arbitrary $C > 0$, $\sigma \neq 0$ and $\bar{x} \in \mathbb{R}^n$.*

Proof. Since u_∞ is a minimizer to (2), we have that

$$E(u_\infty) \leqslant E\left(\frac{u}{\|u\|_p}\right) \leqslant \frac{\|\nabla u\|_2^2}{2\|u\|_p^2} + \frac{\|u\|_q^q}{q\|u\|_p^q}, \quad \forall u \in D^{1,q}(\mathbb{R}^n) \tag{7}$$

with equality if $u = u_\infty$. Scaling u as $u_\lambda(x) = u(\frac{x}{\lambda})$, we have

$$E(u_\infty) \leqslant \min_{\lambda>0}\left[\frac{\lambda^{n-2-\frac{2n}{p}}}{2}\frac{\|\nabla u\|_2^2}{\|u\|_p^2} + \frac{\lambda^{n(1-\frac{q}{p})}}{q}\frac{\|u\|_q^q}{\|u\|_p^q}\right] := \min_{\lambda>0} f(\lambda) \tag{8}$$

for all $0 \neq u \in D^{1,q}(\mathbb{R}^n)$. The minimizer of $f(\lambda)$ is achieved at

$$\lambda_{min} = \left[\frac{2-n+\frac{2n}{p}}{n(1-\frac{q}{p})}\frac{A}{B}\right]^{\frac{p}{2n+2p-nq}}, \quad A = \frac{\|\nabla u\|_2^2}{2\|u\|_p^2}, \ B = \frac{\|u\|_q^q}{q\|u\|_p^q}. \tag{9}$$

Then (8) reads as (1) with the best constant given by (6). Clearly u_∞ is an optimal function of (1). And since (1) is invariant under translation, scaling and multiplication by a constant, then $u_{\sigma,\bar{x}}(x) = Cu_\infty(\sigma(x-\bar{x}))$ are optimal functions of (1), for arbitrary $C > 0$, $\sigma \neq 0$ and $\bar{x} \in \mathbb{R}^n$. □

3. Extremals in Gagliardo–Nirenberg inequalities

This section is devoted to the study of the variational problem (2), and to the computation of an explicit minimizer u_∞ to this problem. Below, we study the existence of a minimizer to (2), and we establish some of its properties which will be needed later.

Theorem 3.1. *Let n, p, q be as in Theorem* 2.1. *Then, the variational problem* (2) *has a minimizer u_∞, which can be chosen non-negative, radially-symmetric, decreasing and tends to* 0 *as $|x|$ tends to ∞. Moreover, u_∞ is the unique radial ground state of the PDE*

$$-\Delta u + u^{q-1} - \lambda u^{p-1} = 0 \tag{10}$$

where $\lambda > 0$ (the Lagrange multiplier) is chosen so that the normalization condition $\|u\|_q = 1$ holds. Therefore the unique radial ground state of (10) (*for a well chosen λ*) *is a minimizer of* (2).

Proof. The existence of a minimizer to (1) follows by compactness. For the properties of the minimizer, we use a rearrangement argument, and for the uniqueness, we refer to Serrin and Tang [10]. □

Now, we establish the ODE leading to the computation of the minimizer u_∞ of problem (2) – i.e., the unique radial ground state of PDE (10) –, and we solve it in general when $n = 1$, and in some particular cases when $n > 1$.

Using the rescaled function $\bar{u}_\infty(x) = \lambda^{\frac{1}{p-q}} u_\infty(\lambda^{\frac{q-2}{2(p-q)}} x)$ in the PDE (10), we have that $\bar{u}_\infty$ is the radial ground state of the PDE $-\Delta u + u^{q-1} - u^{p-1} = 0$. Then there exists a non-negative, decreasing function $v : [0,\infty) \to [0,\infty)$ satisfying $v(\infty) = v'(\infty) = 0$, such that $\bar{u}_\infty(x) = v(r)$, $r := |x|$, and $v(r)$ solves the ODE

$$v''(r) + (n-1)\frac{v'(r)}{r} - v^{q-1}(r) + v^{p-1}(r) = 0, \tag{11}$$

which is equivalent to the previous PDE. Now, using that $v(r) = v \circ g(\frac{r^2}{2})$ where $g(t) = \sqrt{2t}$, that v and g are both invertible, and setting $H = (v \circ g)^{-1}$, we have that

$$H\big(v(r)\big) = \frac{r^2}{2}, \tag{12}$$

and H is decreasing on $(0, v(0))$, with $\lim_{t\to 0^+} H(t) = \infty$ and $\lim_{t\to 0^+} H'(t) = -\infty$ if $q \geqslant 2$, while $0 < \lim_{t\to 0^+} H(t) < \infty$ if $q < 2$ as $v(r)$ has a compact support in this case. The change of function (12) is suggested by the link between certain Gagliardo–Nirenberg inequalities and Mass transportation theory (see details in [2]). The following theorem establishes a first order ODE for H', which leads to the computation of u_∞.

Theorem 3.2. *Let n, p, q be as in Theorem* 2.1. *Let H be defined as in* (12) *where $v(r)$ is a non-negative, decreasing solution of* (11) *such that $v(\infty) = v'(\infty) = 0$. Then $H(t)$ satisfies the non-linear ODE*

$$2\left(\frac{t^q}{q} - \frac{t^p}{p}\right)H''(t) + (t^{q-1} - t^{p-1})H'(t) - 2(n-1)H''(t)\int_0^t \frac{\mathrm{d}s}{H'(s)} = n, \tag{13}$$

for all $t \in (0, v(0))$, and $\lim_{t\to 0^+} H(t) = \infty$ and $\lim_{t\to 0^+} H'(t) = -\infty$ if $q \geqslant 2$, while $0 < \lim_{t\to 0^+} H(t) < \infty$ if $q < 2$.

Proof. First, we differentiate $H(v(r)) = \frac{r^2}{2}$ and combine it with (11) to have that

$$-v''(r) = \frac{n-1}{H'(v(r))} - v^{q-1}(r) + v^{p-1}(r). \tag{14}$$

Then we multiply (14) by $v'(r)$, and integrate over (r, ∞) to obtain that

$$v'(r)^2 = 2\left(\frac{v^q(r)}{q} - \frac{v^p(r)}{p}\right) - 2(n-1)\int_0^{v(r)} \frac{\mathrm{d}s}{H'(s)}. \tag{15}$$

Next, we differentiate $H(v(r)) = \frac{r^2}{2}$ twice with respect to r to get

$$v''(r)H'\big(v(r)\big) + v'(r)^2 H''\big(v(r)\big) = 1. \tag{16}$$

Finally, we insert (14) and (15) into (16) to yield (13) after changing $v(r)$ into t. □

Corollary 3.3. *If $n = 1$ and $1 \leqslant q < p$, then*

$$H'(t) = \frac{p}{2t^{\frac{q}{2}}\sqrt{|\frac{p}{q} - t^{p-q}|}} \int \frac{\mathrm{d}t}{\operatorname{sign}(\frac{p}{q} - t^{p-q})t^{\frac{q}{2}}\sqrt{|\frac{p}{q} - t^{p-q}|}} \tag{17}$$

is a solution of ODE (13)*. Therefore the minimizer u_∞ of problem* (2) *can be computed explicitly via* (17) *and* (12)*, and Theorem* 2.1 *gives the sharp constant, K_{opt}, and the optimal functions, $u_{\sigma,\bar{x}}(x) = Cu_\infty(\sigma(x - \bar{x}))$, of all the Gagliardo–Nirenberg and Nash's inequalities* (1) *when $n = 1$ and $1 \leqslant q < p$.*

Proof. If $n = 1$, (13) reduces to a linear first order ODE in H' whose solution is given by (17). □

Example 1. The minimizer u_∞ of problem (2) is given by,

(i) If $q = 2 < p$, then

$$u_\infty(x) = \left(\frac{p}{2\lambda}\right)^{\frac{1}{p-2}} \left[\cosh\left(\frac{p-2}{2}|x|\right)\right]^{-\frac{2}{p-2}},$$

where λ is determined by $\|u_\infty\|_p = 1$.

(ii) If $1 \leqslant q < p = 2$, then

$$u_\infty(x) = \left(\frac{2}{q\lambda}\right)^{\frac{1}{2-q}} \left[\cos\left(\frac{2-q}{2}|x|\sqrt{\lambda}\right)\right]^{\frac{2}{2-q}} \chi_{[|x| \leqslant \frac{\pi}{(2-q)\sqrt{\lambda}}]}(x).$$

In particular, when $q = 1$ (i.e., L^2-Nash's inequality in dimension $n = 1$), we have

$$u_\infty(x) = \frac{2}{\lambda}\cos^2\left(\frac{|x|\sqrt{\lambda}}{2}\right)\chi_{[|x| \leqslant \frac{\pi}{\sqrt{\lambda}}]}(x) = \frac{1}{\lambda}\big(1 + \cos\big(|x|\sqrt{\lambda}\big)\big)\chi_{[|x| \leqslant \frac{\pi}{\sqrt{\lambda}}]}(x),$$

where λ is determined by $\|u_\infty\|_p = 1$. Note that the sharp constant and extremals of the L^2-Nash's inequality are first obtained by Carlen and Loss in [4].

If $n > 1$, we have not been able to solve (13) in general. But, if we furthermore assume that

$$H''(t)\int_0^t \frac{\mathrm{d}s}{H'(s)} = k = \text{constant}, \tag{18}$$

then (13) becomes again linear, and can be solved explicitly. In this case, we recover the subclass $q = 1 + \frac{p}{2}$ and $q = 2(p-1)$, of the Gagliardo–Nirenberg inequalities obtained by Del-Pino and Dolbeault in [6].

Corollary 3.4. *Under the hypotheses of Theorem* 3.2*, assume that H satisfies* (18)*. Then H solves ODE* (13) *if and only if $q = 1 + \frac{p}{2}$ or $q = 2(p-1)$. Therefore,*

(i) *If $q = 1 + \frac{p}{2}$, $(p > 2)$, then $H(t) = \frac{2(2n-p(n-2))}{(p-2)^2}t^{1-\frac{p}{2}} + \gamma$, for some constant γ, and*

$$u_\infty(x) = \left[\frac{(p-2)^2}{4(2n - p(n-2))}\right]^{\frac{1}{1-p/2}} \left(|x|^2 + \frac{2\lambda(2n - p(n-2))^2}{p(p-2)^2}\right)^{\frac{1}{1-p/2}} \tag{19}$$

is a minimizer of (2)*, where $\lambda > 0$ is uniquely determined by the constraint $\|u_\infty\|_p = 1$.*

(ii) *If $q = 2(p-1)$, $(1 < p < 2)$, then $H(t) = -\frac{2(n-1)-p(n-2)}{(2-p)^2}t^{2-p} + \gamma$, for some constant γ, and*

$$u_\infty(x) = \left[\frac{\lambda(2-p)^2}{2(2(n-1) - p(n-2))}\right]^{\frac{1}{2-p}} \left(\frac{(2(p-1) + n(2-p))^2}{\lambda^2(p-1)(2-p)^2} - |x|^2\right)_+^{\frac{1}{2-p}} \tag{20}$$

is a minimizer of (2)*, where $\lambda > 0$ is uniquely determined by the constraint $\|u_\infty\|_p = 1$.*

In both cases, the best constants K_{opt}*, and optimal functions* $u_{\sigma,\bar{x}}$ *of the corresponding Gagliardo–Nirenberg inequalities* (1) *are given by Theorem* 2.1 *where* u_∞ *is defined by* (19) *or* (20).

Proof. (18) gives that $H'(t) = -At^{\frac{k}{k+1}}$ for some $A > 0$. Inserting this expression into (13) where we first substitute (18), we have that

$$\left(\frac{2k}{q(k+1)} + 1\right)t^{q-\frac{1}{k+1}} - \left(1 + \frac{2k}{p(k+1)}\right)t^{p-\frac{1}{k+1}} = -\frac{n+2(n-1)k}{A}, \quad \forall t \in \big(0, v(0)\big). \tag{21}$$

Since $p \neq q$, (21) holds for all t, if and only if

$$q - \frac{1}{k+1} = 0 \quad \text{and} \quad 1 + \frac{2k}{p(k+1)} = 0, \quad \text{or} \quad 1 + \frac{2k}{q(k+1)} = 0 \quad \text{and} \quad p - \frac{1}{k+1} = 0.$$

We deduce that $q = 1 + \frac{p}{2}$ or $q = 2(p-1)$. (19) and (20) follow easily by integrating $H'(t) = -At^{\frac{k}{k+1}}$ and using (21) and (12). □

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