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Differential Geometry

Uniform tail-decay of Lipschitz functions implies Cheeger's isoperimetric inequality under convexity assumptions ☆

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Abstract

We show that for convex domains in Euclidean space, Cheeger's isoperimetric inequality, spectral-gap of the Neumann Laplacian, exponential concentration of Lipschitz functions, and the a priori weakest uniform tail-decay of these functions, are all equivalent (to within universal constants, independent of the dimension). This substantially extends previous results of Maz'ya, Cheeger, Gromov–Milman, Buser and Ledoux. As an application, we conclude the stability of the spectral-gap for convex domains under convex perturbations which preserve volume (up to constants) and under maps which are "on-average" Lipschitz. We also provide a new characterization of the Cheeger constant, as one over the expectation of the distance from the "worst" Borel set having half the measure of the convex domain. In addition, we easily recover (and extend) many previously known lower bounds, due to Payne–Weinberger, Li–Yau and Kannan–Lovász–Simonovits, on the Cheeger constant of convex domains. Essential to our proof is a result from Riemannian Geometry on the concavity of the isoperimetric profile. Our results extend to the more general setting of Riemannian manifolds with density which satisfy the $CD(0, \infty)$ curvature-dimension condition of Bakry–Émery. ***To cite this article: E. Milman, C. R. Acad. Sci. Paris, Ser. I 346 (2008).***

Résumé

Le queue-affaiblissement uniforme des fonctions lipschitziennes implique l'inégalité isopérimétrique de Cheeger sous hypothèses de convexité. Nous montrons que pour les domaines convexes dans l'espace euclidien, l'inégalité isopérimétrique de Cheeger, l'existence du trou spectral pour le Laplacien de Neumann, la concentration exponentielle des fonctions lipschitziennes et la a priori plus faible propriété de queue-affaiblissement uniforme de ces fonctions, sont toutes équivalentes (à constantes universelles près, indépendamment de la dimension). Ceci étend considérablement des résultats précédents de Maz'ya, Cheeger, Gromov–Milman, Buser et Ledoux. Comme application, nous en déduisons la stabilité du trou spectral des domaines convexes sous perturbations convexes qui préservent le volume (à des constantes près). Nous offrons aussi une nouvelle caractérisation de la constante de Cheeger, comme l'inverse de la moyenne de la distance par rapport au « pire » ensemble borélien ayant la moitié de la mesure du domaine convexe. En outre, nous récupérons facilement (et prolongez) beaucoup de limites inférieures précédemment connues dues à Payne–Weinberger, Li–Yau et Kannan–Lovász–Simonovits, sur la constante de Cheeger des domaines convexes. Nos résultats s'étendent plus généralement aux variétés riemanniennes munies d'une densité qui satisfont la condition de courbure-dimension $CD(0, \infty)$ de Bakry–Émery. ***Pour citer cet article : E. Milman, C. R. Acad. Sci. Paris, Ser. I 346 (2008).***

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Soit (M, g) une variété riemannienne complète de dimension n, d la distance géodésique induite par g et μ une mesure absolument continue par rapport à la forme de volume vol_M de (M, g). Cette Note porte sur le rapport entre la métrique d et la mesure μ.

Rappelons que la mesure de bord (extérieure) de Minkowski d'un sous-ensemble borélien $A \subset M$ est définie par $\mu^+(A) := \liminf_{\varepsilon\to 0} \frac{\mu(A_{\varepsilon,d})-\mu(A)}{\varepsilon}$, où $A_{\varepsilon,d} := \{x \in M;\ \exists y \in A\ d(x, y) < \varepsilon\}$ est l'ε voisinage de A par rapport à la métrique d. L'*inégalite isopérimétrique de Cheeger* affirme l'existence d'une constante $D > 0$ telle que, pour tout sous-ensemble borélien $A \subset M$, $\mu^+(A) \geqslant D \min(\mu(A), 1 - \mu(A))$; nous notons la constante optimale par D_{Che}. Soit D_{Poin} la constante optimale dans l'*inégalité de Poincaré* (voir la Définition 2), de sorte que D_{Poin}^2 est le trou spectral du Laplacien de Neumann Δ_μ associé à la mesure μ. Nous dirons qu'on a *concentration exponentielle* s'il existe une constante $D > 0$ telle que pour toute fonction 1-lipschitzienne f sur (M, d), $\mu(|f - \int f\,\mathrm{d}\mu| \geqslant t) \leqslant e \exp(-Dt)$, et notons la constante optimale par D_{Exp}. Nous dirons enfin qu'on a *concentration du premier moment* s'il existe une constante $D > 0$ telle que pour toute fonction 1-lipschitzienne f sur (M, d), $\int |f - \int f\,\mathrm{d}\mu|\,\mathrm{d}\mu \leqslant 1/D$, et notons la constante optimale par D_{FM}. Cette notion est manifestement plus faible que celle de concentration exponentielle.

Le reste de ces propriétés sont aussi ordonnées hiérarchiquement de la façon suivante. Maz'ya [22] et indépendamment Cheeger [9], ont montré que l'inégalité isopérimétrique de Cheeger implique l'inégalité de Poincaré : $D_{\mathrm{Poin}} \geqslant D_{\mathrm{Che}}/2$ (inégalité de Cheeger). M. Gromov et V. Milman [14] ont montré que l'inégalité de Poincaré implique la concentration exponentielle : il existe une constante universelle $c > 0$ telle que $D_{\mathrm{Exp}} \geqslant c D_{\mathrm{Poin}}$.

Il est facile de voir qu'aucune de ces implications n'est une équivalence en général. Nous ferons par conséquent des hypothèses additionnelles de convexité du type de la condition de courbure-dimension $CD(0, \infty)$ de Bakry–Émery (voir la Définition 5). Celles-ci sont satisfaites par exemple lorsque (M, g) est l'espace euclidien et μ la mesure de probabilié uniforme d'un convexe borné K, auquel cas on noterons la constante isopérimétrique de Cheeger par $D_{\mathrm{Che}}(K)$.

Sous ces hypothèses de convexité, Buser [8], puis plus généralement Ledoux [19], ont montré que l'inégalité de Cheeger peut être renversée : il existe une constante $c > 0$ telle que $D_{\mathrm{Che}} \geqslant c D_{\mathrm{Poin}}$. Notre résultat principal affirme que toutes les autres implications peuvent de même être renversées.

Théorème 0.1. *Sous nos hypothèses de convexité, les propriétés suivantes sont équivalentes* :

(i) *L'inégalite isopérimétrique de Cheeger* (*avec* D_{Che}) ;
(ii) *L'inégalité de Poincaré* (*avec* D_{Poin}) ;
(iii) *La concentration exponentielle* (*avec* D_{Exp}) ;
(iv) *La concentration du premier moment* (*avec* D_{FM}).

L'équivalence est dans le sens que les constantes ci-dessus satisfont $D_{\mathrm{Che}} \simeq D_{\mathrm{Poin}} \simeq D_{\mathrm{Exp}} \simeq D_{\mathrm{FM}}$.

Ici, et plus bas, $A \simeq B$ signifie que $A \leqslant C_1 B \leqslant C_2 A$, où $C_i > 0$ sont des constantes universelles. Le Théorème 0.1 est équivalent à la caractérisation de D_{Che} que nous obtenons dans le Théorème 3.2 ci-dessous. Comme application, nous obtenons le résultat de stabilité suivant (voir la Section 3 et [23] pour d'autres applications) :

Théorème 0.2. *Soient* K, L *deux convexes bornés de l'espace euclidien. Si* $\mathrm{Vol}(K) \simeq \mathrm{Vol}(L) \simeq \mathrm{Vol}(K \cap L)$*, alors* $D_{\mathrm{Che}}(K) \simeq D_{\mathrm{Che}}(L)$.

Nos démonstrations s'appuyent sur les methodés semi-groupe de Ledoux [19] et sur un résultat de géométrie riemannienne sur la concavité du profil isopérimétrique sous nos hypothèses de convexité (Théorème 4.2).

1. Introduction

This Note announces some of the results obtained in [23,24]. Let (Ω, d) denote a separable metric space, and let μ denote a Borel probability measure on (Ω, d). To avoid introducing further definitions, we will in fact assume that Ω is a complete oriented n-dimensional Riemannian manifold (M, g), d is the induced geodesic distance, and μ is an absolutely continuous measure with respect to the Riemannian volume form vol_M on M.

This Note pertains to the interplay between the metric d and the measure μ on the space (Ω, d, μ). One way to measure this relationship is by means of an isoperimetric inequality. Recall that Minkowski's (exterior) boundary measure of a Borel set $A \subset \Omega$, which we denote here by $\mu^+(A)$, is defined as $\mu^+(A) := \liminf_{\varepsilon \to 0} \frac{\mu(A_{\varepsilon,d}) - \mu(A)}{\varepsilon}$, where $A_{\varepsilon,d} := \{x \in \Omega;\ \exists y \in A\ d(x, y) < \varepsilon\}$ denotes the ε extension of A with respect to the metric d. The isoperimetric profile $I = I_{(\Omega,d,\mu)}$ is defined as the pointwise maximal function $I : [0, 1] \to \mathbb{R}_+$, so that $\mu^+(A) \geqslant I(\mu(A))$, for all Borel sets $A \subset \Omega$. For convenience, define $\tilde{I} : [0, 1/2] \to \mathbb{R}_+$ as $\tilde{I}(t) := \min(I(t), I(1 - t))$. An isoperimetric inequality measures the relation between $\mu^+(A)$ and $\mu(A)$ by means of the isoperimetric profile I. A well known example was given by Cheeger [9]:

Definition 1. Our space is said to satisfy Cheeger's isoperimetric inequality if there exists $D > 0$, such that $\tilde{I}(t) \geqslant Dt$ for all $t \in [0, 1/2]$. The best constant D is denoted by $D_{\mathrm{Che}}(\Omega, d, \mu)$.

A second way to measure the interplay between d and μ is given by functional inequalities. Let $\mathcal{F} = \mathcal{F}(\Omega, d)$ denote the space of functions which are Lipschitz on every ball in (Ω, d), and let $f \in \mathcal{F}$. We will consider functional inequalities which measure the relation between $\| f - E_\mu f \|_{N_1(\mu)}$ and $\| |\nabla f| \|_{N_2(\mu)}$, where N_1, N_2 are some norms associated with the measure μ, like the $L_p(\mu)$ norms, and $E_\mu f$ denotes the expectation of f. A well known example is given by Poincaré's inequality:

Definition 2. Our space is said to satisfy Poincaré's inequality if there exists $D > 0$ such that $D\| f - E_\mu(f)\|_{L_2(\mu)} \leqslant \| |\nabla f| \|_{L_2(\mu)}$ for all $f \in \mathcal{F}$. The best constant D is denoted by $D_{\mathrm{Poin}}(\Omega, d, \mu)$.

It is well known that Poincaré's inequality is equivalent to the existence of a spectral-gap of the Laplacian Δ_μ associated to the measure μ with Neumann boundary conditions on its support. The first non-trivial eigenvalue of Δ_μ is then precisely D_{Poin}^2.

A third way to measure the relation between d and μ is given by concentration inequalities. These measure how tightly 1-Lipschitz functions are concentrated about their mean, by providing a quantitative estimate on the tail decay. A typical situation is given by the exponential concentration:

Definition 3. Our space is said to satisfy exponential concentration if there exists $D > 0$ such that for any 1-Lipschitz function f and for all $t > 0$, $\mu(|f - E_\mu(f)| \geqslant t) \leqslant e \exp(-Dt)$. The best constant D is denoted by $D_{\mathrm{Exp}}(\Omega, d, \mu)$.

In this Note, we introduce the following very weak notion of concentration:

Definition 4. Our space is said to satisfy First-Moment concentration if there exists $D > 0$ such that for any 1-Lipschitz function f, $\| f - E_\mu(f)\|_{L_1(\mu)} \leqslant \frac{1}{D}$. The best constant D is denoted by $D_{\mathrm{FM}}(\Omega, d, \mu)$.

Clearly First-Moment concentration implies *linear* tail-decay, and is implied by slightly faster decay, so it is a priori much weaker than exponential concentration.

2. The hierarchy and hierarchy reversal

It is known that the first three examples above are also arranged in a hierarchy. It was shown by V. Maz'ya [22] and independently by Cheeger [9], that Cheeger's isoperimetric inequality always implies Poincaré's inequality: $D_{\mathrm{Poin}} \geqslant D_{\mathrm{Che}}/2$ (Cheeger's inequality). The fact that Poincaré's inequality implies exponential concentration was first shown by M. Gromov and V. Milman [14] in the Riemannian setting, and subsequently by other authors in other settings as well (see e.g. [18]): there exists a universal constant $c > 0$ such that $D_{\mathrm{Exp}} \geqslant c D_{\mathrm{Poin}}$.

It is easy to see that the latter implications cannot be reversed in general, so we will need to impose some further conditions in order to proceed. We start with two important examples when $(M, g) = (\mathbb{R}^n, |\cdot|)$ and $|\cdot|$ is some fixed Euclidean norm:

- μ is the uniform probability measure on a bounded convex domain in $\mathbb{R}^n$ ($n \geqslant 2$).
- μ is an absolutely continuous log-concave probability measure on $\mathbb{R}^n$ ($n \geqslant 1$), meaning that $\mathrm{d}\mu = \exp(-\psi)\,\mathrm{d}x$ where $\psi : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is a convex function.

More generally, we define:

Definition 5. The space (Ω, d, μ) is said to satisfy our convexity assumptions if $\Omega = (M, g)$, d is the induced geodesic distance on M, and $\mathrm{d}\mu = \exp(-\psi)\,\mathrm{d}\,\mathrm{vol}_M$ where $\psi \in C^2(M)$ and as tensor fields $\mathrm{Ric}_g + \mathrm{Hess}_g\,\psi \geqslant 0$ on M (or when μ may be appropriately approximated by such measures).

The latter condition is the well known Curvature-Dimension condition $CD(0, \infty)$, introduced by Bakry and Émery in [1] (in the more abstract framework of diffusion generators). It is known that under our convexity assumptions, Cheeger's inequality can be reversed: $D_{\mathrm{Che}} \geqslant c D_{\mathrm{Poin}}$, where $c > 0$ is a universal constant. This was first shown by Buser [8] when μ is uniform on a closed manifold and $\mathrm{Ric}_g \geqslant 0$, and was recently strengthened and generalized by M. Ledoux [19] to the Bakry–Émery abstract setting. A natural question which arises is whether the Gromov–Milman result can also be reversed under our convexity assumptions. We will see that in fact much more is indeed true.

3. The main results

Theorem 3.1. *Under our convexity assumptions, the following are equivalent for the space* (Ω, d, μ):

(i) *Cheeger's isoperimetric inequality* (*with* D_{Che}); (ii) *Poincaré's inequality* (*with* D_{Poin});
(iii) *Exponential concentration* (*with* D_{Exp}); (iv) *First-Moment concentration* (*with* D_{FM}).

The equivalence is in the sense that the constants above satisfy $D_{\mathrm{Che}} \simeq D_{\mathrm{Poin}} \simeq D_{\mathrm{Exp}} \simeq D_{\mathrm{FM}}$.

Here and below $A \simeq B$ means that $A/C_1 \leqslant B \leqslant C_2 A$, with $C_i > 0$ some (explicit) universal constants (typically not exceeding 100). In fact the use of the First-Moment is not essential in Statement (iv); we may have required *any* fixed uniform tail-decay (with the constants depending on the rate of decay), e.g. control of the rth moment ($L_r(\mu)$) with $r > 0$ instead of $r = 1$. Our Main Theorem may be reformulated as stating that under our convexity assumptions, there exists a single 1-Lipschitz function f of the form $f = d(x, A)$, where A is some Borel set of measure $1/2$, whose level sets *on average* attain the minimum (up to constants) in Cheeger's isoperimetric inequality:

Theorem 3.2. *Under our convexity assumptions,* $D_{\mathrm{Che}}(\Omega, d, \mu) \simeq \inf\{\frac{1}{\int_\Omega d(x,A)\,\mathrm{d}\mu};\ A \subset \Omega,\ \mu(A) \geqslant 1/2\}$.

As our main application, we deduce a stability result for the Cheeger constant $D_{\mathrm{Che}}(K)$ (equivalently, the spectral-gap of the Neumann Laplacian $D^2_{\mathrm{Poin}}(K)$) for bounded domains $K \subset \mathbb{R}^n$ (equipped with the uniform probability distribution on K). Clearly, there can be no stability without some further assumptions, which we add in the form of convexity:

Theorem 3.3. *Let* K, L *denote two bounded convex domains in* $(\mathbb{R}^n, |\cdot|)$. *If* $\mathrm{Vol}(K) \simeq \mathrm{Vol}(L) \simeq \mathrm{Vol}(K \cap L)$*, then* $D_{\mathrm{Che}}(K) \simeq D_{\mathrm{Che}}(L)$.

This implies that when $\frac{1}{a}L \subset K \subset Lb$ with $a, b \geqslant 1$, $ab \leqslant 1 + \frac{c}{n}$, then $D_{\mathrm{Che}}(K) \simeq D_{\mathrm{Che}}(L)$. Sharp quantitative versions of these stability results are given in [23], where their optimality is also demonstrated. To the best of our knowledge, quantitative bounds on the stability of D_{Che} for convex domains under small perturbations were previously unknown. Completely analogous results hold for log-concave probability measures as well. Another useful result we deduce from Theorem 3.1 is that Cheeger's constant is preserved under maps which are not necessarily Lipschitz, but rather Lipschitz on average when the target space satisfies our convexity assumptions, in the sense that

$\int_\Omega \|DT\|_{\mathrm{op}}(x)\,\mathrm{d}\mu(x) \leqslant 1$, where $\|DT\|_{\mathrm{op}}(x)$ denotes the operator norm of the derivative DT at x (local Lipschitz constant).

A conjecture of Kannan, Lovász and Simonovits [16] states that under a natural non-degeneracy condition on a bounded convex domain K, $D_{\mathrm{Che}}(K) \simeq 1$, independently of the dimension n. The upper bound follows from standard Convexity Theory, but the lower bound is far from being resolved. There are many known lower bounds which provide dimension dependent results, and we are able to easily recover many of them, without appealing to the localization method used by KLS (originating in the work of Gromov–Milman [15]). These include results by Payne and Weinberger [27], Li and Yau [20] and KLS [16]. For instance, we show that the following KLS bound:

$$D_{\mathrm{Che}}(\Omega, d, \mu) \geqslant \sup_{x_0 \in \Omega} \frac{c}{\int d(x, x_0)\,\mathrm{d}\mu(x)},$$

for some universal constant $c > 0$, remains valid on an arbitrary Riemannian manifold with density satisfying our convexity assumptions, whereas the localization method is confined to Euclidean space (and a few other special manifolds). Using Theorem 3.1, we also recover a lower bound on D_{Che} due to S. Bobkov [6] and a bound on D_{Che} for the ℓ_p^n balls, for $p \in [1, 2]$, due to S. Sodin [28].

4. Outline of proof

The convexity assumptions are used in an essential way in the proof of the Main Theorem in two separate places. First, we employ the $CD(0, \infty)$ condition via the semi-group gradient estimates used by Ledoux in [19]. Contrary to previous approaches, which could only deduce isoperimetric information from functional inequalities with a $\||\nabla f|\|_{L_q(\mu)}$ term with $q = 2$, we can handle arbitrary $q \geqslant 1$.

To demonstrate that our estimates are sharp, we show that the isoperimetric inequalities we obtain are in fact equivalent (up to universal constants) to the functional inequalities used to derive them. This is achieved using the language of capacities, which are certain functional formulations of isoperimetric inequalities, introduced around 1960 by Maz'ya [21] and Federer and Fleming [10]. We call a convex increasing function $N : \mathbb{R}_+ \to \mathbb{R}_+$ so that $N(0) = 0$ a Young function, and associate to it as usual the Orlicz-norm $N(\mu)$.

Definition 6. The space (Ω, d, μ) is said to satisfy an Orlicz–Sobolev inequality (with constant $D > 0$) if for any $f \in \mathcal{F}$, $D\|f - E_\mu f\|_{N(\mu)} \leqslant \||\nabla f|\|_{L_q(\mu)}$.

Theorem 4.1. *Let $1 \leqslant q \leqslant \infty$, and let N denote a Young function, so that*:

$$N(t)^{1/q}/t \text{ is non-decreasing}, \quad \exists \alpha > (1/q - 1/2) \vee 0 \text{ so that } N(t^\alpha)/t \text{ is non-increasing}. \qquad (*)$$

Then under our convexity assumptions, the following statements are equivalent:

(i) *Orlicz–Sobolev inequality* (*with constant* D_1); (ii) $\tilde{I}(t) \geqslant D_2 t^{1-1/q}/N^{-1}(1/t)$ *for all* $t \in [0, 1/2]$, *where the best constants D_1, D_2 above satisfy $c_1 C_{\alpha,q} D_1 \leqslant D_2 \leqslant c_2 B_{\alpha,q} D_1$, with $c_1, c_2 > 0$ universal constants and $B_{\alpha,q}, C_{\alpha,q}$ depending explicitly on α, q. In fact, the convexity assumptions are not needed for the direction* (ii) $\Rightarrow$ (i), *and the assumptions* $(*)$ *are not needed if $q \geqslant 2$ for the direction* (i) $\Rightarrow$ (ii).

When $N(t) = t^2$, $q = 2$, the direction (ii) $\Rightarrow$ (i) reduces (up to constants) to Cheeger's inequality, and the direction (i) $\Rightarrow$ (ii) to Ledoux's generalization of Buser's Theorem. In addition, using $N(t) = t^q \log(1 + t^q)$ and a result of Bobkov and Zegarlinski [7], a slight variant of Theorem 4.1 implies that q-log-Sobolev inequalities ($1 \leqslant q \leqslant 2$) are equivalent to the isoperimetric inequality $\tilde{I}(t) \geqslant ct \log^{1/q}(1/t)$ under our convexity assumptions. The case $q = 2$ was previously shown by Ledoux [19].

Applying Theorem 4.1, we deduce from the First-Moment inequality ($N(t) = t$, $q = \infty$) that $\tilde{I}(t) \geqslant c D_{\mathrm{FM}} t^2$ for $t \in [0, 1/2]$. To deduce Cheeger's inequality from this, we need to use our convexity assumptions for the second time. We employ the following result in Riemannian Geometry (in fact, we additionally employ an approximation argument to handle the case of a non-smooth domain or density):

Theorem 4.2. *(See [2,11,26,29,17,4,3,25,5].) Under our convexity assumptions, the isoperimetric profile $I = I_{(\Omega,d,\mu)}$ is concave on* $(0, 1)$. *In fact, when μ is uniform on $\Omega \subset (M, g)$, where M is an n-dimensional manifold* (*and our convexity assumptions are satisfied*), *then $I^{n/(n-1)}$ is concave on* $[0, 1]$.

To deduce this theorem, various results provided by Geometric Measure Theory on the existence and regularity of isoperimetric minimizers are needed; a similar approach was used by M. Gromov ([12], [13, Appendix C]) in his generalization of P. Lévy's isoperimetric inequality on the sphere. The concavity of I (and its symmetry about the point $1/2$) immediately imply that under out convexity assumptions $D_{\mathrm{Che}} = 2I(1/2)$. Consequently, we deduce from $\tilde{I}(t) \geqslant c D_{\mathrm{FM}} t^2$ that $D_{\mathrm{Che}} \geqslant \frac{c}{2} D_{\mathrm{FM}} > 0$, concluding the proof of the Main Theorem. It is clear that a stronger statement can be deduced when μ is uniform on Ω (using the concavity of $I^{n/(n-1)}$).

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