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Partial Differential Equations

Correlation between two quasilinear elliptic problems with a source term involving the function or its gradient

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Abstract

Thanks to a change of unknown we compare two elliptic quasilinear problems with Dirichlet data in a bounded domain of $\mathbb{R}^N$. The first one, of the form $-\Delta_p u = \beta(u)|\nabla u|^p + \lambda f(x)$, where β is nonnegative, involves a gradient term with natural growth. The second one, of the form $-\Delta_p v = \lambda f(x)(1+g(v))^{p-1}$ where g is nondecreasing, presents a source term of order 0. The correlation gives new results of existence, nonexistence and multiplicity for the two problems. ***To cite this article: H.A. Hamid, M.F. Bidaut-Véron, C. R. Acad. Sci. Paris, Ser. I 346 (2008).***

Résumé

Corrélation entre deux problèmes quasilinéaires elliptiques avec terme de source relatif à la fonction ou à son gradient. A l'aide d'un changement d'inconnue nous comparons deux problèmes elliptiques quasilinéaires avec conditions de Dirichlet dans un domaine borné Ω de $\mathbb{R}^N$. Le premier, de la forme $-\Delta_p u = \beta(u)|\nabla u|^p + \lambda f(x)$, où β est positif, comporte un terme de gradient à croissance critique. Le second, de la forme $-\Delta_p v = \lambda f(x)(1+g(v))^{p-1}$ où g est croissante, contient un terme de source d'ordre 0. La comparaison donne des résultats nouveaux d'existence, nonexistence et multiplicité pour les deux problèmes. ***Pour citer cet article : H.A. Hamid, M.F. Bidaut-Véron, C. R. Acad. Sci. Paris, Ser. I 346 (2008).***

Version française abrégée

Soit Ω un domaine borné régulier de $\mathbb{R}^N$ $(N \geqslant 2)$ et $1 < p < N$. Dans cette Note nous comparons deux problèmes quasilinéaires. Le premier comporte un terme de source d'ordre 1 :

$$-\Delta_p u = \beta(u)|\nabla u|^p + \lambda f(x) \quad \text{dans } \Omega, \qquad u = 0 \quad \text{sur } \partial\Omega, \tag{1}$$

où $\beta \in C^0([0, L))$, $L \leqslant \infty$, à valeurs $\geqslant 0$, $\lambda > 0$ et $f \in L^1(\Omega)$, $f \geqslant 0$ p.p. dans Ω. Le second problème comporte un terme de source d'ordre 0 :

$$-\Delta_p v = \lambda f(x)\big(1 + g(v)\big)^{p-1} \quad \text{dans } \Omega, \qquad v = 0 \quad \text{sur } \partial\Omega, \tag{2}$$

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où $g \in C^1([0,\Lambda))$, $\Lambda \leqslant \infty$, $g(0)=0$ et g est croissante.

Le changement d'inconnue

$$v(x)=\Psi\big(u(x)\big)=\int_0^{u(x)} e^{\gamma(\theta)/(p-1)}\,d\theta, \quad \text{où } \gamma(t)=\int_0^t \beta(\theta)\,d\theta,$$

conduit formellement du problème (1) au problème (2), et β et g sont liés par la relation $\beta(u)=(p-1)g'(v)$. En particulier β est croissant si et seulement si g est convexe. Le changement d'inconnue inverse formel, apparemment moins utilisé, est donné explicitement par

$$u(x)=H\big(v(x)\big)=\int_0^{v(x)} \frac{ds}{1+g(s)}.$$

Toutefois dans la transformation peuvent s'introduire des mesures. Notons $M_b^+(\Omega)$ l'espace des mesures de Radon positives bornées sur Ω, et $M_s^+(\Omega)$ le sous-ensemble des mesures concentrées sur un ensemble de p-capacité 0. Nous établissons une correspondance précise entre les deux problèmes :

Théorème 1. *Soit u une solution renormalisée du problème*

$$-\Delta_p u=\beta(u)|\nabla u|^p+\lambda f(x)+\alpha_s \quad \text{dans } \Omega, \qquad u=0 \quad \text{sur } \partial\Omega, \tag{3}$$

où $\alpha_s \in M_s^+(\Omega)$, et $0 \leqslant u(x) < L$ p.p. dans Ω. Alors il existe $\mu_s \in M_b^+(\Omega)$, telle que $v=\Psi(u)$ est solution atteignable du problème

$$-\Delta_p v=\lambda f(x)\big(1+g(v)\big)^{p-1}+\mu_s \quad \text{dans } \Omega, \qquad v=0 \quad \text{sur } \partial\Omega. \tag{4}$$

Réciproquement soit v une solution renormalisée de (4), telle $0 \leqslant v(x) < \Lambda$ p.p. dans Ω, où $\mu_s \in M_s^+(\Omega)$. Alors il existe $\alpha_s \in M_s^+(\Omega)$, telle que $u=H(v)$ est solution renormalisée de (3). De plus, si $\mu_s=0$, alors $\alpha_s=0$. Si $L=\infty$ et $\beta \in L^1((0,\infty))$, alors $\mu_s=e^{\gamma(\infty)}\alpha_s$. Si $L<\infty$, ou $L=\infty$ et $\beta \notin L^1((0,\infty))$, et $\alpha_s \neq 0$ alors (3) n'a pas de solution. Si $\Lambda<\infty$ et $\mu_s \neq 0$, alors (4) n'a pas de solution.

Dans le cas β constant, les résultats suivants généralisent ceux de [1] relatifs au cas $p=2$:

Théorème 2. *On suppose que $\beta(u) \equiv p-1$, donc $v=\Psi(u)=e^u-1$ et $g(v)=v$, et que*

$$\lambda_1(f)=\inf\left\{\left(\int_\Omega |\nabla w|^p\,dx\right)\Big/\left(\int_\Omega f|w|^p\,dx\right):\ w \in W_0^{1,p}(\Omega)\setminus\{0\}\right\}>0.$$

Si $\lambda>\lambda_1(f)$, ou $\lambda=\lambda_1(f)$ et $f \in L^{N/p}(\Omega)$, alors (1) et (2) n'ont pas de solution renormalisée.

Si $0<\lambda<\lambda_1(f)$ alors (2) a une solution unique $v_0 \in W_0^{1,p}(\Omega)$, et (1) a une solution unique $u_0 \in W_0^{1,p}(\Omega)$ telle que $e^{u_0}-1 \in W_0^{1,p}(\Omega)$. Si de plus $f \in L^r(\Omega)$ avec $r>N/p$, alors u_0 et $v_0 \in L^\infty(\Omega)$; et pour toute mesure $\mu_s \in M_s^+(\Omega)$, (4) a une solution renormalisée v_s, et donc (1) a une infinité de solutions $u_s=H(v_s) \in W_0^{1,p}(\Omega)$ moins régulières que u_0.

Le Théorème 1 et l'utilisation du problème (1) nous permettent de déduire un résultat important pour le problème (2), étendant un résultat classique de [2] dans le cas $p=2$:

Théorème 3. *On suppose que $\Lambda=\infty$, $\lim_{s\to\infty} g(s)/s=\infty$, g est convexe à l'infini, et $f \in L^r(\Omega)$ avec $r>N/p$. Alors il existe $\lambda^*>0$ tel que pour tout $\lambda \in (0,\lambda^*)$ le problème (2) a une solution minimale bornée $\underline{v}_\lambda$, et pour tout $\lambda>\lambda^*$ il n'a aucune solution renormalisée.*

Nous étudions aussi les propriétés de la fonction extrémale $v^*=\sup_{\lambda\nearrow\lambda^*}\underline{v}_\lambda$ étendant certains résultats de [3,9,11]. Dans le cas où g est à croissance limitée par l'exposant de Sobolev, nous obtenons des résultats d'existence d'une seconde solution variationnelle, nouveaux même dans le cas $p=2$, étendant ceux de [1] et de [5].

1. Introduction and main results

Let Ω be a smooth bounded domain in $\mathbb{R}^N$ $(N \geqslant 2)$ and $1 < p < N$. In this Note we compare two quasilinear problems. The first one presents a source gradient term with a natural growth:

$$-\Delta_p u = \beta(u)|\nabla u|^p + \lambda f(x) \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega, \tag{1}$$

where $\beta \in C^0([0, L))$, $L \leqslant \infty$, and β is nonnegative, and $\lambda > 0$ is a given real, and $f \in L^1(\Omega)$, $f \geqslant 0$ a.e. in Ω. Here β can have an asymptote at point L, and is not supposed to be nondecreasing.

The second problem involves a source term of order 0, with the same λ and f:

$$-\Delta_p v = \lambda f(x)\big(1 + g(v)\big)^{p-1} \quad \text{in } \Omega, \qquad v = 0 \quad \text{on } \partial\Omega, \tag{2}$$

where $g \in C^1([0, \Lambda))$, $\Lambda \leqslant \infty$, $g(0) = 0$ and g is nondecreasing.

Problems of type (1) and (2) have been intensively studied the last twenty years. The main questions are existence, according to the regularity of f and the value of λ, regularity and multiplicity of the solutions, and the existence with possible measure data.

It is well known that the change of unknown in (1)

$$v(x) = \Psi\big(u(x)\big) = \int_0^{u(x)} \mathrm{e}^{\gamma(\theta)/(p-1)}\,\mathrm{d}\theta, \quad \text{where } \gamma(t) = \int_0^t \beta(\theta)\,\mathrm{d}\theta,$$

leads formally to problem (2), where $\Lambda = \Psi(L)$ and g is given by $g(v) = \mathrm{e}^{\gamma(u)/(p-1)} - 1$. This is a way for studying problem (1) from the knowledge of problem (2). It is apparently less used the reverse correlation, even in case $p = 2$: for *any* function g nondecreasing on $[0, \Lambda)$, the substitution in (2)

$$u(x) = H\big(v(x)\big) = \int_0^{v(x)} \frac{\mathrm{d}s}{1 + g(s)}$$

leads formally to problem (1), where β is defined on $[0, L)$ with $L = H(\Lambda)$; indeed $H = \Psi^{-1}$. And β is linked to g by relation $\beta(u) = (p-1)g'(v)$. In particular β is nondecreasing if and only if g is convex; and L is finite if and only if $1/(1+g) \notin L^1(0, \Lambda)$.

Some examples with $p = 2$.

1. $\beta(u) = 1$ and $1 + g(v) = 1 + v$.
2. $\beta(u) = q/(1 + (1-q)u)$, $q \in (0, 1)$, and $1 + g(v) = (1 + v)^q$.
3. $\beta(u) = 1 + \mathrm{e}^u$ and $1 + g(v) = (1 + v)(1 + \ln(1 + v))$.
4. $\beta(u) = q/(1 - (q-1)u)$, $q > 1$ and $1 + g(v) = (1 + v)^q$.
5. $\beta(u) = 1/(1 - u)$ and $1 + g(v) = \mathrm{e}^v$.
6. $\beta(u) = q/(1 - (q+1)u)$, $q > 0$ and $1 + g(v) = 1/(1 - v)^q$.

It had been observed in [6] that the correspondence between u and v is more complex, because some measures can appear, in particular in the equation in v. Our first main result is to make precise the link between the two problems. We denote by $M_b(\Omega)$ the set of bounded Radon measures, and by $M_s(\Omega)$ the subset of measures concentrated on a set of p-capacity 0. And $M_b^+(\Omega)$ and $M_s^+(\Omega)$ are the positive cones of $M_b(\Omega)$, $M_s(\Omega)$, and $M_0(\Omega)$ is the subset of measures absolutely continuous with respect to the p-capacity. Recall that $M_b(\Omega) = M_0(\Omega) + M_s(\Omega)$.

We recall one definition of renormalized solutions among four of them given in [4]. Let U be measurable and finite a.e. in Ω, such that $T_k(U)$ belongs to $W_0^{1,p}(\Omega)$ for any $k > 0$. One still denotes by u the cap_p-quasi-continuous representative. Let $\mu = \mu_0 + \mu_s^+ - \mu_s^- \in M_b(\Omega)$. Then U is a renormalized solution of problem

$$-\Delta_p U = \mu \quad \text{in } \Omega, \qquad U = 0 \quad \text{on } \partial\Omega, \tag{3}$$

if $|\nabla U|^{p-1} \in L^\tau(\Omega)$, for any $\tau \in [1, N/(N-1))$, and for any $k > 0$, there exist $\alpha_k, \beta_k \in M_0(\Omega) \cap M_b^+(\Omega)$, concentrated on the sets $\{U = k\}$ and $\{U = -k\}$ respectively, converging in the narrow topology to μ_s^+, μ_s^- such that for any $\psi \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$,

$$\int_\Omega \left|\nabla T_k(U)\right|^{p-2} \nabla T_k(U) \cdot \nabla \psi \, \mathrm{d}x = \int_{\{|U|<k\}} \psi \, \mathrm{d}\mu_0 + \int_\Omega \psi \, \mathrm{d}\alpha_k - \int_\Omega \psi \, \mathrm{d}\beta_k.$$

Theorem 1.1. *Let u be any renormalized solution of problem*

$$-\Delta_p u = \beta(u)|\nabla u|^p + \lambda f(x) + \alpha_s \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega, \tag{4}$$

where $\alpha_s \in M_s^+(\Omega)$ and such that $0 \leqslant u(x) < L$ a.e. in Ω. Then there exists $\mu_s \in M_b^+(\Omega)$, such that $v = \Psi(u)$ is a reachable solution of problem

$$-\Delta_p v = \lambda f(x)\big(1 + g(v)\big)^{p-1} + \mu_s \quad \text{in } \Omega, \qquad v = 0 \quad \text{on } \partial\Omega. \tag{5}$$

Conversely let v be any renormalized solution of (5)*, where $\mu_s \in M_s^+(\Omega)$ and such that $0 \leqslant v(x) < \Lambda$ a.e. in Ω. Then there exists $\alpha_s \in M_s^+(\Omega)$ such that $u = H(v)$ is a renormalized solution of* (4).

Moreover if $\mu_s = 0$, then $\alpha_s = 0$. If $L = \infty$ and $\beta \in L^1((0,\infty))$, then $\mu_s = \mathrm{e}^{\gamma(\infty)}\alpha_s$. If $L < \infty$ or if $L = \infty$ and $\beta \notin L^1((0,\infty))$, and $\alpha_s \neq 0$, then (4) *has no solution. If $\Lambda < \infty$, and $\mu_s \neq 0$, then* (5) *has no solution.*

This theorem extends in particular the results of [1] where $p = 2$ and $L = \infty$. The nonexistence result when $\beta \notin L^1((0,\infty))$, and $\alpha_s \neq 0$, answers to an open question of [10].

First we apply to the case β constant, which means g linear.

Theorem 1.2. *Assume that $\beta(u) \equiv p - 1$, thus $v = \Psi(u) = \mathrm{e}^u - 1$ and $g(v) = v$. Suppose that*

$$\lambda_1(f) = \inf\left\{\left(\int_\Omega |\nabla w|^p \, \mathrm{d}x\right) \Big/ \left(\int_\Omega f|w|^p \, \mathrm{d}x\right): \; w \in W_0^{1,p}(\Omega) \setminus \{0\}\right\} > 0. \tag{6}$$

If $\lambda > \lambda_1(f)$, or $\lambda = \lambda_1(f)$ and $f \in L^{N/p}(\Omega)$, then (1) *and* (2) *admit no renormalized solution.*

If $0 < \lambda < \lambda_1(f)$ there exists a unique solution $v_0 \in W_0^{1,p}(\Omega)$ to (2)*, thus a* unique *solution $u_0 \in W_0^{1,p}(\Omega)$ to* (1) *such that $\mathrm{e}^{u_0} - 1 \in W_0^{1,p}(\Omega)$. If $f \in L^r(\Omega)$ with $r > N/p$, then u_0 and $v_0 \in L^\infty(\Omega)$, and moreover for any measure $\mu_s \in M_s^+(\Omega)$, there exists a renormalized solution v_s of* (5)*; then there exists an* infinity *of less regular solutions $u_s = H(v_s) \in W_0^{1,p}(\Omega)$ of* (1).

Remark 1.3. Under the assumption (6), most of these existence results extend to general g such that $\Lambda = \infty$ and $\limsup_{\tau \longrightarrow \infty} g(\tau)/\tau < \infty$. They extend to the case

$$\limsup_{\tau \to \infty} g(\tau)/\tau^q < \infty \quad \text{for some } q \in \big(1, N/(N-p)\big) \tag{7}$$

if moreover $f \in L^r(\Omega)$ with $qr' < N/(N-p)$.

Next consider problem (2) with general g, and $f \in L^r(\Omega)$ with $r > N/p$. It is easy to prove that for small $\lambda > 0$ there exists a minimal solution $\underline{v}_\lambda \in W_0^{1,p}(\Omega)$ such that $\|\underline{v}_\lambda\|_{L^\infty(\Omega)} < \Lambda$. Our main result is an extension of a well known result of [2] for $p = 2$, and [3] for $p > 1$. It is noteworthy that *the proof uses problem* (1):

Theorem 1.4. *Assume that $\Lambda = \infty$, and $\lim_{s\to\infty} g(s)/s = \infty$, and g is convex near infinity, and $f \in L^r(\Omega)$ with $r > N/p$. There exists a real $\lambda^* > 0$ such that*:

(i) *for $\lambda \in (0, \lambda^*)$ problem* (2) *has a minimal* bounded *solution $\underline{v}_\lambda$,*
(ii) *for $\lambda > \lambda^*$ there exists no* renormalized *solution.*

When g is subcritical with respect to the Sobolev exponent $p^* = Np/(N-p)$, we obtain *new multiplicity results* for problem (2), even in the case $p = 2$, extending [1] and [5]:

Theorem 1.5. *Under the assumptions of Theorem* 1.4*, assume that*

$$\limsup_{\tau\to\infty} g^{p-1}(\tau)/\tau^{Q} < \infty \quad \text{for some } Q \in (1, p^* - 1), \tag{8}$$

and $f \in L^r(\Omega)$ *with* $(Q+1)r' < p^*$*. Then there exists* $\lambda_0 > 0$ *such that for any* $\lambda < \lambda_0$*, there exists at least* two solutions $v \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ *of* (2)*. Moreover if* $p = 2$*,* g *is convex, or* g *satisfies the Ambrosetti–Rabinowitz growth condition and* $f \in L^\infty(\Omega)$*, the same result holds with* $\lambda_0 = \lambda^*$*.*

Concerning the extremal solution, we get the following, extending some results of [3,11]:

Theorem 1.6. *Under the assumptions of Theorem* 1.4*, the extremal function* $v^* = \sup_{\lambda \nearrow \lambda^*} \underline{v}_\lambda$ *is a renormalized solution of* (2) *with* $\lambda = \lambda^*$*. If* $N < p(1+p')/(1+p'/r)$ *then* $v^* \in W_0^{1,p}(\Omega)$*. Moreover* $v^* \in L^\infty(\Omega)$ *in any of the following conditions*:

(i) N *is arbitrary and* (8) *holds and* $(Q+1)r' < p^*$*,*
(ii) N *is arbitrary and* (7) *holds and* $qr' < N/(N-p)$*,*
(iii) $N < pp'/(1 + 1/(p-1)r)$*.*

Remark 1.7. Using Theorems 1.1, 1.4 and 1.5, we deduce existence and nonexistence results for problem (1). In Theorem 1.1, function f can depend on u or v, which strongly extends the range of applications. For example, taking $g(v) = v$, and $f = u^b$, $b > 0$, problem $-\Delta_p u = (p-1)|\nabla u|^p + \lambda u^b$ relative to u leads to $-\Delta_p v = \lambda(1+v)^{p-1}\ln^b(1+v)$ relative to v. Then for small λ the problem in u has an infinity of solutions $u \in W_0^{1,p}(\Omega)$, two of them being bounded.

Remark 1.8. A part of our results is based on a growth assumption on g. Returning to problem (1), this condition is not always easy to verify. When $L = \infty$, all the "usual" functions β, even with a strong growth, satisfy $\limsup_{\tau\to\infty} g(\tau)/\tau^q < \infty$ for any $q > 0$, see [1]. However using the converse correlation between g and β, we prove that the conjecture that this condition always holds is *wrong*: let $F \in C^0([0,\infty))$ be any strictly convex function, with $\lim_{s\to\infty} F(s) = \infty$. Then there exists an increasing function β such that $\lim_{t\to\infty}\beta(t) = \infty$ and the corresponding g satisfies $\limsup_{\tau\to\infty} g(\tau)/F(\tau) = \infty$.

2. Sketch of the main proofs

In some proofs we use a regularity lemma:

Lemma 2.1. *Let* $1 < p < N$*, and* $F \in L^m(\Omega)$*, and* $\overline{m} = Np/(Np - N + p)$ (*thus* $1 < \overline{m} < N/p$)*. Let* U *be a renormalized solution of problem*

$$-\Delta_p U = F \quad \text{in } \Omega, \qquad U = 0 \quad \text{on } \partial\Omega.$$

If $1 < m < N/p$*, then* $U^{(p-1)} \in L^k(\Omega)$*, with* $k = Nm/(N-pm)$*. If* $m = N/p$*, then* $U^{(p-1)} \in L^k(\Omega)$ *for any* $k \geqslant 1$*. If* $m > N/p$*, then* $U \in L^\infty(\Omega)$*. If* $1 < m < \overline{m}$*, then* $|\nabla U|^{(p-1)} \in L^\tau(\Omega)$*, with* $\tau = Nm/(N-m)$*. If* $m \geqslant \overline{m}$*, then* $U \in W_0^{1,p}(\Omega)$*.*

Proof of Theorem 1.1. For $p \neq 2$, we cannot use approximations of the equations because of the nonuniqueness of the solutions of $-\Delta_p U = \mu$ with $\mu \in M_b^+(\Omega)$. The main idea is to use the equations satisfied in the sense of distributions by the truncated functions $T_K(u) = \min(K, u)$ and $T_k(v) = \min(k, v)$ with $k = \Psi(K)$, using definition (ii) of renormalized solution given above:

$$-\Delta_p T_K(u) = \beta\big(T_K(u)\big)\big|\nabla T_K(u)\big|^p + \lambda f \chi_{\{u\leqslant K\}} + \alpha_K, \quad \text{in } \mathcal{D}'(\Omega),$$
$$-\Delta_p T_k(v) = \lambda f\big(1+g(v)\big)^{p-1}\chi_{\{v\leqslant k\}} + \mu_k, \quad \text{in } \mathcal{D}'(\Omega),$$

where μ_k and α_K are two measures concentrated on the same set: $\{u = K\} = \{v = k\}$, and explicitly related by $\mu_k = (1+g(k))^{p-1}\alpha_K$, and respectively converging weakly* to μ_s and α_s. The nonexistence results are consequences of some properties of renormalized solutions, also called Inverse Maximum Principle. □

Proof of Theorem 1.2. The nonexistence is first proved for (1), and then for (2) by Theorem 1.1. The existence is obtained by iteration and approximation, using [4]. Uniqueness follows from Picone's identity, adapted to renormalized solutions. □

Proof of Theorem 1.4. Formally, if v is a solution of (2) for some λ, and $u = H(v)$, then $\bar{u} = (1-\varepsilon)u$ is a supersolution of (1) relative to $\bar{\lambda} = (1-\varepsilon)^{p-1}\lambda$, and $\bar{v} = \Psi(\bar{u})$ is a supersolution of (2) relative to $\bar{\lambda}$; then there exists a solution $v_1 \leqslant \bar{v}$. Using Theorem 1.1 we show that it is not formal, since actually *no measure appears*. In the (best) case $H(\infty) < \infty$, $\bar{v}$ is bounded, then also v_1 is bounded. Otherwise a bootstrapp using Lemma 2.1 is needed for constructing a bounded solution. □

Proof of Theorem 1.5. The Euler function J_λ is well defined, and for small λ it has the geometry of Mountain Path near 0, but the Palais–Smale sequences may be unbounded. From [8], (2) has a second solution for almost any small λ, and then for a sequence $\lambda_n \to \lambda$, and we are lead to prove that the solutions v_{λ_n} relative to λ_n converge to a solution to (2) relative to λ. The usual estimates for the case $p = 2$, using an eigenfunction as test function, cannot be extended. We get an estimate of $-\Delta_p v_{\lambda_n}$ in $L^1(\Omega)$ in another way, using the convexity of g. The estimate of v_{λ_n} in $W_0^{1,p}(\Omega)$ is obtained by contradiction. For larger λ, if $p = 2$, J_λ has still the geometry of mountain path near $\underline{v}_\lambda$; the question is open when $p \neq 2$. Under Ambrosetti–Rabinowitz condition we apply some results of [7]. □

Proof of Theorem 1.6. The estimates come from Lemma 2.1 and well known regularity results for quasilinear equations, and from an extension of techniques of [9]. □

References

[1] B. Abdellaoui, A. Dall'Aglio, I. Peral, Some remarks on elliptic problems with critical growth in the gradient, J. Differential Equations 222 (2006) 21–62.
[2] H. Brezis, T. Cazenave, Y. Martel, A. Ramiandrisoa, Blow-up for $u_t - \Delta u = g(u)$ revisited, Adv. Differential Equations 1 (1996) 73–90.
[3] X. Cabre, M. Sanchon, Semi-stable and extremal solutions of reaction equations involving the p-Laplacian, Comm. Pure Appl. Anal. 6 (2007) 43–67.
[4] G. Dal Maso, F. Murat, L. Orsina, A. Prignet, Renormalized solutions of elliptic equations with general measure data, Ann. Scuola Norm. Sup. Pisa 28 (1999) 741–808.
[5] A. Ferrero, On the solutions of quasilinear elliptic equations with a polynomial-type reaction term, Adv. Differential Equations 9 (2004) 1201–1234.
[6] V. Ferone, F. Murat, Nonlinear problems having natural growth in the gradient: an existence result when the source terms are small, Nonlinear Anal. 42 (2000) 1309–1326.
[7] N. Ghoussoub, D. Preiss, A general mountain path principle for locating and classifying critical points, Ann. Inst. H. Poincaré Anal. Non Linéaire 6 (5) (1989) 321–330.
[8] L. Jeanjean, On the existence of bounded Palais–Smale sequences and application to a Landesman–Lazer type problem, Proc. Roy. Soc. Edinburgh Sect. A 129 (1999) 787–809.
[9] G. Nedev, Regularity of the extremal solution of semilinear elliptic equations, C. R. Acad. Sci. Paris Sér. I Math. 330 (2000) 997–2002.
[10] A. Porretta, Nonlinear equations with natural growth terms and measure data, Electron. J. Differential Equations Conf. 9 (2002) 183–202.
[11] M. Sanchon, Boundedness of the extremal solutions of some p-Laplacian problems, Nonlinear Anal. 67 (2007) 281–294.