Optimal Control

# An Ingham type proof for the boundary observability of a $N - d$ wave equation

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**Abstract**

The boundary observability of the wave equation has been studied by many authors. A method of choice is to use the multiplier method (cf. Komornik (1994)). Recently, in Ramdani et al. (2005), a first Fourier based proof is given in the case where the domain is a square, thanks to a new Hautus type test. We give here a new self-contained proof with an Ingham type approach in the more general case where the domain is a product of intervals; this leads, in contrary to the proof in Ramdani et al., to explicit time and constants. However, we do not reach the optimal time which can be obtained for this problem by the multiplier method. ***To cite this article: M. Mehrenberger, C. R. Acad. Sci. Paris, Ser. I 347 (2009).***

**Résumé**

**Une preuve de type Ingham pour l'observabilité frontière d'une équation des ondes $N - d$.** L'observabilité frontière de l'équation des ondes a été étudiée par de nombreux auteurs. Une méthode de choix est d'utiliser la méthode des multiplicateurs (cf. Komornik (1994)). Récemment, dans Ramdani et al. (2005), une première preuve basée sur les séries de Fourier a été donnée dans le cas où le domaine est un carré grâce à un test de type Hautus. On donne ici une nouvelle preuve auto-contenue par une approche de type Ingham, dans le cas plus général où le domaine est un produit d'intervalles ; on obtient alors un temps et des constantes explicites, contrairement à la preuve de Ramdani et al. (2005). Cependant, on n'atteint pas le temps optimal, qui peut être obtenu pour ce problème par la méthode des multiplicateurs. ***Pour citer cet article : M. Mehrenberger, C. R. Acad. Sci. Paris, Ser. I 347 (2009).***

## Version française abrégée

Soit $N \in \mathbb{N}^*$, des réels $L_i > 0$, $i = 1, \ldots, N$, $\Omega = \prod_{i=1}^{N}]0, L_i[$ et $u(t, x)$ solution de (1). Le système est bien défini pour des conditions initiales satisfaisant $(u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega)$. On a la proposition classique suivante :

**Proposition 0.1.** *Il existe $T_0 > 0$ tel que pour $T > T_0$, le système* (1) *est observable* : *il existe une constante $c > 0$ telle que l'on ait* (2) *pour tout $(u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega)$.*

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La méthode des multiplicateurs (cf. [3]) permet d'obtenir cette inégalité d'observabilité (2) avec un temps optimal $T_0 = 2\sqrt{\sum_{i=1}^{N} L_i^2}$.

Dans certaines situations, on peut aussi essayer d'obtenir une preuve utilisant une inégalité d'Ingham (cf. [1, 4]). Comme la solution de (1) est explicitement donnée en termes de série de Fourier, une question naturelle est : peut-on obtenir (2) avec des arguments de type Ingham ? La difficulté principale, dans le cadre multi-dimensionnel ($N > 1$) est que des valeurs propres en nombre arbitraire peuvent être arbitrairement proches, ce qui rend l'obtention de l'inégalité d'observabilité par une telle approche plus difficile. Notons que de tels cas ont déjà été considérés dans d'autres situations (voir [6,5]). Certains théorèmes multi-dimensionnels de type Ingham ont été établis et ont permis de traiter par exemple du cas de l'équation des plaques avec contrôle interne [2]. Notons que (2) a été prouvé pour le cas du carré à l'aide de séries de Fourier, grâce à un test de type Hautus dans [7], sans temps explicite d'observation. Nous mentionnons que l'étude de l'inégalité d'observabilité peut aussi être effectuée par plusieurs autres techniques, telles que l'analyse micro-locale ou les estimations de Carleman. Le but de cette présente Note est de fournir une preuve auto-contenue de type Ingham de (2), dans le cas (déjà précisé) où le domaine est un produit d'intervalles. On utilise d'abord l'orthogonalité des vecteurs propres, puis on applique la première méthode d'Ingham. Grâce à une décomposition convenable, on peut se débarasser des termes où il n'y a pas d'écart spectral (comme dans [6]) et on peut donc obtenir l'inégalité d'observabilité désirée. Le temps d'observation, ainsi que les constantes impliquées dans l'inégalité peuvent être explicitées, mais nous n'avons pas pu atteindre le temps optimal par cette méthode.

## 1. Introduction

Let $N \in \mathbb{N}^*$, reals $L_i > 0$, $i = 1, \ldots, N$, $\Omega = \prod_{i=1}^{N}]0, L_i[$ and $u(t, x)$ solution of

$$\begin{cases} u'' = \Delta u, & 0 < t < T, \quad x \in \Omega, \\ u = 0, & 0 < t < T, \quad x \in \partial\Omega, \\ u(0, x) = u_0(x), \quad u'(0, x) = u_1(x), \quad x \in \Omega. \end{cases} \tag{1}$$

The system is well defined for initial conditions satisfying $(u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega)$. We have the following classical proposition:

**Proposition 1.1.** Let $\Gamma = \bigcup_{j=1}^{N} (\prod_{i=1}^{j-1}[0, L_i] \times \{0\} \times \prod_{i=j+1}^{N}[0, L_i])$. There exists $T_0 > 0$ such that for $T > T_0$, the system (1) is observable: there exists a constant $c > 0$ such that we have

$$\|u_0\|_{H_0^1(\Omega)}^2 + \|u_1\|_{L^2(\Omega)}^2 \leqslant c \int_0^T \int_\Gamma |\partial_\nu u(t, x)|^2 \, \mathrm{d}\Gamma \, \mathrm{d}t, \tag{2}$$

for all $(u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega)$.

One famous method to study such an observability inequality is to use the multiplier method (cf. [3]). We can here effectively apply this method and obtain (2) with an optimal time $T_0 = 2\sqrt{\sum_{i=1}^{N} L_i^2}$.

In some situations, we can also try to use an Ingham inequality (cf. [1,4]).

The solution of (1) is explicitly given by terms of a Fourier series:

$$u(t, x) = \sum_{k \in (\mathbb{N}^*)^N} (\alpha_k \mathrm{e}^{\mathrm{i}\omega_k t} + \alpha_{-k} \mathrm{e}^{-\mathrm{i}\omega_k t}) \prod_{j=1}^{N} \sin\left(\frac{\pi k_j x_j}{L_j}\right), \tag{3}$$

with suitable coefficients $\alpha_k$, and with the notations

$$k = (k_1, \ldots, k_N), \quad x = (x_1, \ldots, x_N), \qquad \omega_k = \sqrt{\sum_{j=1}^{N} \left(\frac{k_j \pi}{L_j}\right)^2}.$$

A natural question is then, can we obtain (2) with Ingham type arguments ? The main difficulty, in the multi-dimensional setting ($N > 1$), is that arbitrarily many eigenvalues can be arbitrarily close, which makes the obtention of

the observability inequality by such an approach more difficult. Note that such cases have already been considered in other situations (see [6,5]). Some multi-dimensional Ingham type theorems have been stated and enabled to deal with e.g. the case of the plate equation with internal control [2]. Note that (2) has been proven for the case of the square by means of Fourier series, thanks to a Hautus type test in [7], without an explicit time of observation. We mention that the study of the observability inequality can also be done by several other techniques, for instance microlocal tools or Carleman estimates. The aim of the present note is to provide a simple self contained Ingham type proof of (2), in the case (already precised) where the domain is a product of intervals. We first use the orthogonality of the eigenfunctions, and then apply Ingham's first method. Thanks to a convenient decomposition, we can get rid of the terms where there is no gap (like in [6]) and thus obtain the desired observability inequality. The time of observation, like the constant involved in the inequality can be explicated, but we were not able to reach the optimal time with that method.

In the following, we will write $A \asymp B$ instead of $c_1 A \leqslant B \leqslant c_2 A$ and $A \succsim B$ instead of $A \geqslant c_1 B$ for brevity, where $c_1, c_2 > 0$ are constants.

## 2. A new Ingham inequality

We present here a new Ingham inequality which will enable us to prove Proposition 1.1 as an application, and which may have its own interest. It also could be applied for other problems.

**Theorem 2.1.** *Let $d \in \mathbb{N}^*$, reals $(\lambda_k)_{k \in (\mathbb{N}^*)^d}$, and complexes $(p_\ell)_{\ell \in \mathbb{N}^*}$.*
*We suppose the following gap assumption: for $j = 1, \ldots, d$, there exists $\gamma_j > 0$ such that*

$$|\lambda_{k_1,\ldots,k_{j-1},k_j,k_{j+1},\ldots,k_d} - \lambda_{k_1,\ldots,k_{j-1},k'_j,k_{j+1},\ldots,k_d}| \geqslant \gamma_j |k_j - k'_j|, \tag{4}$$

$$|\lambda_{k_1,\ldots,k_{j-1},k_j,k_{j+1},\ldots,k_d} + \lambda_{k_1,\ldots,k_{j-1},k'_j,k_{j+1},\ldots,k_d}| \geqslant \gamma_j |k_j + k'_j|, \tag{5}$$

*for all the indices $k = (k_1, \ldots, k_d) \in (\mathbb{N}^*)^d$ and $k'_j \in \mathbb{N}^*$ such that the weights $(p_\ell)_{\ell \in \mathbb{N}^*}$ satisfy*

$$\max_{i=1,\ldots,d,\ i \neq j} |p_{k_i}| \leqslant \max(|p_{k_j}|, |p_{k'_j}|). \tag{6}$$

*Then, for $T > 2\pi \sqrt{\sum_{i=1}^{d}(1/\gamma_i^2)}$, there exists a constant $c_1 > 0$ such that we have,*

$$\sum_{j=1}^{d} \sum_{k_1,\ldots,k_{j-1},k_{j+1},\ldots,k_d \in \mathbb{N}^*} \int_0^T \left| \sum_{k_j \in \mathbb{N}^*} p_{k_j} \left( \beta_k e^{i\lambda_k t} + \beta_{-k} e^{-i\lambda_k t} \right) \right|^2 dt$$

$$\geqslant c_1 \sum_{k \in (\mathbb{N}^*)^d} \left( |\beta_k|^2 + |\beta_{-k}|^2 \right) \left( \sum_{j=1}^{d} |p_{k_j}|^2 \right), \tag{7}$$

*for all complexes $(\beta_k)_{k \in (\mathbb{N}^*)^d}$ and $(\beta_{-k})_{k \in (\mathbb{N}^*)^d}$, such that the sums involved are finite.*

The proof consists in using Ingham's first method. We recall here the definition and the main properties that will be useful. We consider the function

$$f(t) = \begin{cases} \cos \frac{\pi t}{T} & \text{if } |t| \leqslant T/2, \\ 0 & \text{if } |t| > T/2. \end{cases}$$

Its Fourier transform $\hat{f}$ satisfies: $\hat{f}(x) = \int_{-\infty}^{\infty} f(t) e^{ixt} \, dt = -(2T\pi \cos(xT/2))/(x^2 T^2 - \pi^2)$. We have $\hat{f}(0) = \frac{2T}{\pi}$, and for $\gamma > \frac{2\pi}{T}$, $\ell \in \mathbb{N}^*$ and $|x| \geqslant \ell\gamma$,

$$\left| \hat{f}(x) \right| \leqslant \frac{2T}{\pi} \frac{1}{|x^2 T^2/\pi^2 - 1|} = \frac{2T}{\pi} \left( \frac{2\pi}{\gamma T} \right)^2 \frac{1}{|4(x/\gamma)^2 - (2\pi/\gamma T)^2|} \leqslant \frac{2T}{\pi} \left( \frac{2\pi}{\gamma T} \right)^2 \frac{1}{4\ell^2 - 1}. \tag{8}$$

We recall also that

$$\sum_{\ell=1}^{\infty} \frac{1}{4\ell^2 - 1} = \frac{1}{2} \sum_{\ell=1}^{\infty} \left( \frac{1}{2\ell - 1} - \frac{1}{2\ell + 1} \right) = \frac{1}{2}. \tag{9}$$

We set $b_k(t) = \beta_k e^{i\lambda_k t} + \beta_{-k} e^{-i\lambda_k t}$. Now, since $f(t) \leqslant 1_{[-T/2, T/2]}$, we have

$$A := \int_{-T/2}^{T/2} \left| \sum_{k_j \in \mathbb{N}^*} p_{k_j} b_k(t) \right|^2 \mathrm{d}t \geqslant \int_{\mathbb{R}} f(t) \left| \sum_{k_j \in \mathbb{N}^*} p_{k_j} b_k(t) \right|^2 \mathrm{d}t.$$

We note $\delta_j = \max_{i=1,\dots,d,\ i \neq j} |p_{k_i}|$ and then have the decomposition

$$A \geqslant \int_{\mathbb{R}} f(t) \left| \sum_{k_j \in \mathbb{N}^*,\ \delta_j \leqslant |p_{k_j}|} p_{k_j} b_k(t) + \sum_{k_j \in \mathbb{N}^*,\ \delta_j > |p_{k_j}|} p_{k_j} b_k(t) \right|^2 \mathrm{d}t$$

$$= \sum_{(k_j, k_j') \in \Lambda_1} + \sum_{(k_j, k_j') \in \Lambda_2} + \sum_{(k_j, k_j') \in \Lambda_3} B_k + \int_{\mathbb{R}} f(t) \left| \sum_{k_j \in \mathbb{N}^*,\ \delta_j > |p_{k_j}|} p_{k_j} b_k(t) \right|^2 \mathrm{d}t, \tag{10}$$

by setting for brevity $k' = (k_1, \dots, k_{j-1}, k_j', k_{j+1}, \dots, k_d)$, $\Lambda_1 := \{(k_j, k_j') \in \mathbb{N}^*,\ \delta_j \leqslant |p_{k_j}|, \delta_j \leqslant |p_{k_j'}|\}$,

$$\Lambda_2 := \{(k_j, k_j') \in \mathbb{N}^*,\ \delta_j \leqslant |p_{k_j}|,\ \delta_j > |p_{k_j'}|\},$$

$$\Lambda_3 := \{(k_j, k_j') \in \mathbb{N}^*,\ \delta_j \leqslant |p_{k_j}|,\ \delta_j > |p_{k_j'}|\},$$

and, since $\hat{f}(x) = \hat{f}(-x)$, $B_k = p_{k_j} \overline{p_{k_j'}}((\beta_k \overline{\beta_{k'}} + \beta_{-k} \overline{\beta_{-k'}}) \hat{f}(\lambda_k - \lambda_{k'}) + (\beta_k \overline{\beta_{-k'}} + \beta_{-k} \overline{\beta_{k'}}) \hat{f}(\lambda_k + \lambda_{k'}))$. The key point here is that we can get rid of the last term of (10) where there is no gap assumption thanks to the positiveness of $f$. This trick was already used in [6] (see also [5]).

For the first term of (10), by using the assumption (4)–(6), the fact that $T > \frac{2\pi}{\gamma_j}$, the property (8), and since $\hat{f}(0) = \frac{2T}{\pi}$, we get, by putting $A_k = |\beta_k|^2 + |\beta_{-k}|^2$,

$$\sum_{(k_j, k_j') \in \Lambda_1} B_k \geqslant \frac{2T}{\pi} \left( \sum_{k_j \in \mathbb{N}^*,\ \delta_j \leqslant |p_{k_j}|} |p_{k_j}|^2 A_k - \sum_{(k_j, k_j') \in \Lambda_1,\ k_j \neq k_j'} C_k \right),$$

where

$$C_k = \frac{|p_{k_j}|^2 A_k + |p_{k_j'}|^2 A_{k'}}{2} \left( \frac{2\pi}{T\gamma_j} \right)^2 \left( \frac{1}{4(k_j - k_j')^2 - 1} + \frac{1}{4(k_j + k_j')^2 - 1} \right).$$

For the second and the third terms of (10), we obtain similarly

$$\sum_{(k_j, k_j') \in \Lambda_2} B_k \geqslant -\frac{2T}{\pi} \sum_{(k_j, k_j') \in \Lambda_2} C_k, \qquad \sum_{(k_j, k_j') \in \Lambda_3} B_k \geqslant -\frac{2T}{\pi} \sum_{(k_j, k_j') \in \Lambda_3} C_k.$$

By gathering the estimates of the four terms of (10) and then using (9), we thus get

$$\frac{\pi}{2T} A \geqslant \sum_{k_j \in \mathbb{N}^*,\ \delta_j \leqslant |p_{k_j}|} |p_{k_j}|^2 |\beta_k|^2 - \sum_{k_j, k_j' \in \mathbb{N}^*,\ k_j \neq k_j'} C_k$$

$$\geqslant \sum_{k_j \in \mathbb{N}^*,\ \delta_j \leqslant |p_{k_j}|} |p_{k_j}|^2 |\beta_k|^2 - \sum_{k_j \in \mathbb{N}^*} |\beta_k p_{k_j}|^2 \left( \frac{2\pi}{T\gamma_j} \right)^2.$$

We then obtain, by summing over $j = 1, \dots, d$ and $k \in (\mathbb{N}^*)^d$,

$$B := \sum_{j=1}^{d} \sum_{k_1,\ldots,k_{j-1},k_{j+1},\ldots,k_d \in \mathbb{N}^*} \int_{-T/2}^{T/2} \left| \sum_{k_j \in \mathbb{N}^*} p_{k_j} \left( \beta_k e^{i\lambda_k t} + \beta_{-k} e^{-i\lambda_k t} \right) \right|^2 dt$$

$$\succsim \sum_{j=1}^{d} \sum_{k \in (\mathbb{N}^*)^d, \, \delta_j \leqslant |p_{k_j}|} A_k |p_{k_j}|^2 - \sum_{j=1}^{d} \left( \frac{2\pi}{T\gamma_j} \right)^2 \sum_{k \in (\mathbb{N}^*)^d} A_k |p_{k_j}|^2$$

$$= \sum_{k \in (\mathbb{N}^*)^d} A_k \sum_{j=1}^{d} \left( 1_{\delta_j \leqslant |p_{k_j}|} - \left( \frac{2\pi}{T\gamma_j} \right)^2 \right) |p_{k_j}|^2.$$

Since we notice that $\sum_{j=1}^{d} 1_{\delta_j \leqslant |p_{k_j}|} |p_{k_j}|^2 \geqslant \max_{j=1,\ldots,d} |p_{k_j}|^2 \geqslant \sum_{j=1}^{d} |\frac{p_{k_j}}{\gamma_j}|^2 / \sum_{j=1}^{d} \frac{1}{\gamma_j^2}$, we finally get

$$B \succsim \left( 1/\sum_{j=1}^{d} \frac{1}{\gamma_j^2} - \left( \frac{2\pi}{T} \right)^2 \right) \sum_{k \in (\mathbb{N}^*)^d} A_k \left( \sum_{j=1}^{d} \left| \frac{p_{k_j}}{\gamma_j} \right|^2 \right) \succsim \left( 1/\sum_{j=1}^{d} \frac{1}{\gamma_j^2} - \left( \frac{2\pi}{T} \right)^2 \right) \sum_{k \in (\mathbb{N}^*)^d} A_k \left( \sum_{j=1}^{d} |p_{k_j}|^2 \right),$$

which ends the proof of Theorem 2.1, from the assumption $T > 2\pi \sqrt{\sum_{i=1}^{d} \frac{1}{\gamma_i^2}}$ and since we can change the interval $[-T/2, T/2]$ into $[0, T]$ by a classical translation argument.

## 3. Application for the proof of Proposition 1.1

In order to prove Proposition 1.1 by using Theorem 2.1, we first express the inequality (2) in terms of the Fourier series (3). We have at first

$$\int_0^T \int_\Gamma |\partial_\nu u(t,x)|^2 \, d\Gamma \, dt = \sum_{j=1}^{N} \int_0^T \int_0^{L_1} \cdots \int_0^{L_{j-1}} \int_0^{L_{j+1}} \cdots \int_0^{L_N} \left| \sum_{k \in (\mathbb{N}^*)^N} \frac{\pi k_j}{L_j} \left( \alpha_k e^{i\omega_k t} + \alpha_{-k} e^{-i\omega_k t} \right) \right.$$

$$\left. \times \prod_{\ell=1, \, \ell \neq j}^{N} \sin \left( \frac{\pi k_\ell x_\ell}{L_\ell} \right) \right|^2 dx_1 \cdots dx_{j-1} \, dx_{j+1} \, dx_N \, dt.$$

Due to the orthogonality of the family $(\sin(\frac{\pi k_\ell x_\ell}{L_\ell}))_{k_\ell \geqslant 1}$ in $L^2(0, L_\ell)$, we get

$$\int_0^T \int_\Gamma |\partial_\nu u(t,x)|^2 \, d\Gamma \, dt \asymp \sum_{j=1}^{N} \sum_{k_1,\ldots,k_{j-1},k_{j+1},\ldots,k_N \in \mathbb{N}^*} \int_0^T \left| \sum_{k_j \in \mathbb{N}^*} k_j \left( \alpha_k e^{i\omega_k t} + \alpha_{-k} e^{-i\omega_k t} \right) \right|^2 dt. \tag{11}$$

On the other hand, we have

$$\|u_0\|_{H_0^1(\Omega)}^2 + \|u_1\|_{L^2(\Omega)}^2 \asymp \sum_{k \in (\mathbb{N}^*)^N} \left( \sum_{j=1}^{N} |k_j|^2 \right) (|\alpha_k|^2 + |\alpha_{-k}|^2). \tag{12}$$

We then apply Theorem 2.1: we take $d = N$, $\lambda_k = \omega_k = \sqrt{\sum_{j=1}^{N} (k_j \pi / L_j)^2}$, for $k \in (\mathbb{N}^*)^N$ and $p_\ell = \ell$, for $\ell \in \mathbb{N}^*$.

In order to check the gap assumption (4)–(6), for a fixed $j \in \{1, \ldots, N\}$, we thus consider $k = (k_1, \ldots, k_N) \in (\mathbb{N}^*)^N$ and $k_j' \in \mathbb{N}^*$, such that

$$0 < k_i \leqslant \max(k_j, k_j'), \quad i = 1, \ldots, N, \, i \neq j, \tag{13}$$

and we have to obtain (4), (5), for a suitable $\gamma_j$. We have at first $|\omega_k + \omega_{k'}| \geqslant |k_j + k_j'| \frac{\pi}{L_j}$. We then compute

$$|\omega_k - \omega_{k'}| = |k_j - k_j'| \frac{\pi^2}{L_j^2} \frac{|k_j + k_j'|}{\sqrt{D + (k_j \pi / L_j)^2} + \sqrt{D + (k_j' \pi / L_j)^2}},$$

with $D = \sum_{i=1,\ i \neq j}^{N} (k_i \pi / L_i)^2$. We can suppose for instance that $k_j < k'_j$ and thus have by using (13)

$$\frac{|\omega_k - \omega_{k'}|}{|k_j - k'_j|} \geqslant \frac{\pi^2}{L_j^2} \frac{|1 + (k_j/k'_j)|}{\sqrt{\sum_{i=1,\ i \neq j}^{d} (\pi/L_i)^2 + (k_j/k'_j)^2 \pi^2 / L_j^2} + \sqrt{\sum_{i=1}^{d} (\pi/L_i)^2}}$$

$$\geqslant \frac{\pi^2}{(L_j)^2} \frac{1}{2\sqrt{\sum_{i=1}^{d} (\pi/L_i)^2}}.$$

We get thus (4), (5) by taking $\gamma_j = (\pi^2 / L_j^2) \frac{1}{2\sqrt{\sum_{i=1}^{d} (\pi/L_i)^2}} < \pi / L_j$. From (11) and (12), we finally get (2) with

$$T > 2\pi \sqrt{\sum_{j=1}^{N} \gamma_j^2} = 4 \sqrt{\sum_{j=1}^{N} (L_j)^4} \sqrt{\sum_{j=1}^{N} L_j^{-2}},$$

which ends the proof of Proposition 1.1. Note that if we could take $\gamma_j = \pi / L_j$, we would obtain the optimal time $T_0 = 2\sqrt{\sum_{i=1}^{N} L_i^2}$.

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