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Differential Geometry

Concentration and isoperimetry are equivalent assuming curvature lower bound

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Abstract

It is well known that isoperimetric inequalities imply in a very general measure-metric-space setting appropriate concentration inequalities. The former bound the boundary measure of sets as a function of their measure, whereas the latter bound the measure of sets separated from sets having half the total measure, as a function of their mutual distance. We show that under a lower bound condition on the Bakry–Émery curvature tensor of a Riemannian manifold equipped with a density, completely general concentration inequalities imply back their isoperimetric counterparts, up to dimension *independent* bounds. Our method is entirely geometric, continuing the approach of Gromov, Buser and Morgan. ***To cite this article: E. Milman, C. R. Acad. Sci. Paris, Ser. I 347 (2009).***

Résumé

Concentration et isopérimétrie sont équivalentes lorsque la courbure est bornée inférieurement. Il est connu que les inégalités isopérimétriques impliquent dans un cadre espace-mesure-métrique très général des inégalités de concentration correspondantes. Les premières bornent la mesure de bord d'un ensemble en fonction de sa mesure, tandis que les dernières bornent la mesure d'un ensemble qui est séparé d'un ensemble ayant la moitié de la mesure totale, en fonction de leur distance mutuelle. Nous montrons que sous une condition de borne inférieure sur le tenseur de courbure de Bakry–Émery d'une variété riemannienne équipée d'une densité, les inégalités de concentration complètement générales impliquent inversement leurs analogues isopérimétriques, à des constantes près qui sont *indépendantes* de la dimension. Notre méthode est entièrement géométrique, continuant l'approche de Gromov, Buser et Morgan. ***Pour citer cet article : E. Milman, C. R. Acad. Sci. Paris, Ser. I 347 (2009).***

1. Introduction

Let (Ω, d) denote a separable metric space, and let μ denote a Borel probability measure on (Ω, d). One way to measure the interplay between the metric d and the measure μ is by means of an isoperimetric inequality. Recall that Minkowski's (exterior) boundary measure of a Borel set $A \subset \Omega$, which we denote here by $\mu^+(A)$, is

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defined as $\mu^+(A) := \liminf_{\varepsilon \to 0} \frac{\mu(A^d_\varepsilon) - \mu(A)}{\varepsilon}$, where $A^d_\varepsilon := \{x \in \Omega;\ \exists y \in A\ d(x, y) < \varepsilon\}$ denotes the ε extension of A with respect to the metric d. The isoperimetric profile $\mathcal{I} = \mathcal{I}_{(\Omega,d,\mu)}$ is defined as the pointwise maximal function $\mathcal{I} : [0, 1] \to \mathbb{R}_+$, so that $\mu^+(A) \geqslant \mathcal{I}(\mu(A))$, for all Borel sets $A \subset \Omega$. An isoperimetric inequality measures the relation between the boundary measure and the measure of a set, by providing a lower bound on $\mathcal{I}_{(\Omega,d,\mu)}$ by some function $J : [0, 1] \to \mathbb{R}_+$ which is not identically 0. Since A and $\Omega \setminus A$ will typically have the same boundary measure, it will be convenient to also define $\tilde{\mathcal{I}} : [0, 1/2] \to \mathbb{R}_+$ as $\tilde{\mathcal{I}}(v) := \min(\mathcal{I}(v), \mathcal{I}(1 - v))$.

Another way to measure the relation between d and μ is given by concentration inequalities. The log-concentration profile $\mathcal{K} = \mathcal{K}_{(\Omega,d,\mu)}$ is defined as the pointwise maximal function $\mathcal{K} : \mathbb{R}_+ \to \mathbb{R}$ such that $1 - \mu(A^d_r) \leqslant \exp(-\mathcal{K}(r))$ for all Borel sets $A \subset \Omega$ with $\mu(A) \geqslant 1/2$. Note that $\mathcal{K}(r) \geqslant \log 2$ for all $r \geqslant 0$. Concentration inequalities measure how tightly the measure μ is concentrated around sets having measure $1/2$ as a function of the distance r away from these sets, by providing a lower bound on $\mathcal{K}$ by some increasing function $\alpha : \mathbb{R}_+ \to \mathbb{R} \cup \{+\infty\}$, so that α tends to infinity. The two main differences between isoperimetric and concentration inequalities are that the latter ones only measure the concentration around sets having measure $1/2$, and do not provide any information for small distances r (smaller than $R_\alpha := \alpha^{-1}(\log 2)$). We refer to [10] for a wider exposition on these and related topics.

It is known and easy to see that an isoperimetric inequality implies a concentration inequality, simply by "integrating" along the isoperimetric differential inequality. For instance, if $\gamma : [\log 2, \infty) \to \mathbb{R}_+$ is a continuous function, it is an easy exercise to show (e.g. [15, Proposition 1.7]) that:

$$\tilde{\mathcal{I}}(v) \geqslant v\gamma(\log 1/v) \quad \forall v \in [0, 1/2] \quad \Rightarrow \quad \mathcal{K}(r) \geqslant \alpha(r) \quad \forall r > 0 \quad \text{where} \quad \alpha^{-1}(x) = \int_{\log 2}^{x} \frac{\mathrm{d}y}{\gamma(y)}. \tag{1}$$

The converse statement, that a concentration inequality implies an isoperimetric inequality, is certainly false in general. This is especially apparent when considering a space (Ω, d) with bounded diameter R, in which case $\mathcal{K}(r) = +\infty$ for all $r > R$; but if the support of μ is disconnected, we can have $\mathcal{I}(v_i) = 0$ for any finite collection of points $\{v_i\} \subset (0, 1)$. Of course, this type of counterexample may also be achieved without demanding that the support of μ be disconnected, but rather by forcing μ to have little mass ("necks") in between massive regions.

We will henceforth assume that Ω is a complete oriented n-dimensional Riemannian manifold (M, g), that d is the induced geodesic distance (when $n = 1$ we will simply consider the Euclidean space $(\mathbb{R}, |\cdot|)$), and that μ is an absolutely continuous measure with respect to the Riemannian volume form vol_M on M. The above examples demonstrate that in order to have any chance of showing that concentration inequalities imply isoperimetric ones, we need to impose some further conditions which would prevent the existence of small necks. It is therefore very natural, at least intuitively, to require lower bounds on some appropriate curvatures of (M, g) and μ. The purpose of this note is to announce that in this case, concentration indeed implies isoperimetry, with quantitative estimates which *do not depend* on the dimension n of M.

2. Results

Definition 1. We will say that our κ-semi-convexity assumptions are satisfied ($\kappa \geqslant 0$) if $\mathrm{d}\mu = \exp(-\psi)\,\mathrm{d}\,\mathrm{vol}_M$ where $\psi \in C^2(M)$, and as tensor fields on M, $\mathrm{Ric}_g + \mathrm{Hess}_g\,\psi \geqslant -\kappa g$ (or when μ may be appropriately approximated by such measures). When $\kappa = 0$, we will say that our convexity assumptions are satisfied.

Here Ric_g denotes the Ricci curvature tensor of (M, g) and Hess_g denotes the second covariant derivative. $\mathrm{Ric}_g + \mathrm{Hess}_g\,\psi$ is the well-known Bakry–Émery curvature tensor, introduced in [1] (in the more abstract framework of diffusion generators), which incorporates the curvature from both the geometry of (M, g) and the measure μ. When ψ is sufficiently smooth, our κ-semi-convexity assumption is then precisely the Curvature-Dimension condition $CD(-\kappa, \infty)$ (see [1]).

Theorem 1. *Under our convexity assumptions*:

$$\mathcal{K}(r) \geqslant \alpha(r) \quad \forall r \geqslant R_\alpha \quad \Rightarrow \quad \tilde{\mathcal{I}}(v) \geqslant \min\bigl(cv\gamma(\log 1/v), c_\alpha\bigr) \quad \forall v \in [0, 1/2] \quad \textit{where} \quad \gamma(x) = \frac{x}{\alpha^{-1}(x)}, \tag{2}$$

where $c > 0$ is a universal numeric constant and $c_\alpha > 0$ is a constant depending solely on α.

Theorem 1 may be generalized as follows:

Theorem 2. *For any $\kappa \geqslant 0$ and increasing function $\alpha : \mathbb{R}_+ \to \mathbb{R} \cup \{+\infty\}$ so that*:

$$\exists r_0 \geqslant 0\ \forall r \geqslant r_0 \quad \alpha(r) \geqslant \delta_0 \kappa r^2 \tag{3}$$

for some universal numeric constant $\delta_0 > 0$, we have under our κ-semi-convexity assumptions:

$$\mathcal{K}(r) \geqslant \alpha(r) \quad \forall r \geqslant R_\alpha \quad \Rightarrow \quad \tilde{\mathcal{I}}(v) \geqslant \min\bigl(c\, v \gamma(\log 1/v), c_{\alpha,\kappa}\bigr) \quad \forall v \in [0, 1/2] \quad \textit{where} \quad \gamma(x) = \frac{x}{\alpha^{-1}(x)},$$

where $c > 0$ is a universal numeric constant and $c_{\alpha,\kappa}$ is a constant depending solely on α and κ.

Since the value of $\alpha(r)$ for $r < R_\alpha$ is irrelevant for both assumption and conclusion in these theorems, one may replace R_α in both theorems by 0; our present formulation emphasizes that it is only the tail behaviour of α which is of importance. We also remark that the constants $c, \delta_0, c_\alpha, c_{\alpha,\kappa}$ above have explicit values and formulas we will provide in [12].

3. Optimality

Up to these constants, Theorem 1 is an almost optimal counterpart to (1). For instance, under the assumptions of Theorem 1, we may obtain from (1) and (2) that for all $p \geqslant 1$:

$$\tilde{\mathcal{I}}(v) \geqslant v \log^{1-\frac{1}{p}} \frac{1}{v} \quad \Rightarrow \quad \mathcal{K}(r) \geqslant \left(\frac{r}{p} + \log^{\frac{1}{p}} 2\right)^p \quad \Rightarrow \quad \tilde{\mathcal{I}}(v) \geqslant \frac{c}{p} v \log^{1-\frac{1}{p}} \frac{1}{v}, \tag{4}$$

where $c > 0$ is a universal constant. A typical model for these inequalities is obtained by considering $(\mathbb{R}, |\cdot|)$ equipped with the probability measure $\exp(-|x/s_p|^p)\,\mathrm{d}x$, where $s_p > 0$ is a scaling factor (our convexity assumptions are indeed satisfied in this case). Note the deterioration in p in the conclusion of (4), which is especially apparent in the limit as $p \to \infty$, where we have under the same assumptions as above:

$$\tilde{\mathcal{I}}(v) \geqslant v \log \frac{1}{v} \quad \Rightarrow \quad \mathcal{K}(r) \geqslant (\log 2) \exp(r) \quad \Rightarrow \quad \tilde{\mathcal{I}}(v) \geqslant c v \frac{\log(1/v)}{\log\log(2/v)}, \tag{5}$$

where $c > 0$ is a universal constant. This phenomenon also appears in our previous joint work with Sasha Sodin [15], where the expression $x/\alpha^{-1}(x)$ (appearing in (2)) also naturally appears (as in other works too, e.g. [4]); this will be further investigated in [12]. In any case, it is easy to show that the extra $\log\log 2/v$ factor appearing in (5) is the worst possible gap one can obtain by this procedure.

We also remark that the condition (3) on α in Theorem 2 is necessary when $\kappa > 0$ (up to the value of the numeric constant δ_0 in (3)) even in the one-dimensional case. This follows from an example of Chen and Wang [8], improving a previous example of Wang [18]. For any $\kappa > 0$ and $0 < \delta < 1/2$, these authors constructed a measure $\mu = \exp(-\psi(x))\,\mathrm{d}x$ on $(\mathbb{R}_+, |\cdot|)$ such that $\psi'' \geqslant -\kappa$ and $\mathcal{K}(r) \geqslant \delta\kappa r^2 + c_{\delta,\kappa}$ for all $r > 0$, and yet $(\mathbb{R}_+, |\cdot|, \mu)$ does not satisfy a log-Sobolev inequality, and hence (in fact, equivalently by [2,11]), it does not satisfy a Gaussian isoperimetric inequality: $\liminf_{v\to 0} \tilde{\mathcal{I}}(v)/v\sqrt{\log 1/v} = 0$. Moreover, it follows from [18] that $1/2$ is an upper bound on the value of δ in any possible counter example as above. It is easy to generalize this construction to arbitrary dimension, and this demonstrates that the conclusion of Theorem 2 cannot hold without requiring that (3) holds with $\delta_0 \geqslant 1/2$.

4. Previously known results

Several variants of Theorems 1 and 2, where our concentration assumption is replaced by the weaker assumption that $\int_\Omega \exp(\alpha(d(x_0, x)))\,\mathrm{d}\mu(x) < \infty$ for some (any) fixed $x_0 \in \Omega$, were previously considered by various authors, primarily for the case $\alpha(r) = \delta r^p$ $(p \geqslant 1)$. Unfortunately, the constants in the conclusion of these previous results always depended on the quantity $\int d(x_0, x)\,\mathrm{d}\mu$, which under suitable normalizations (e.g. $\max_{x\in\Omega} \mathrm{d}\mu/\mathrm{d}\,\mathrm{vol}_M(x) = 1$) may be shown to be at least as large as a dimension dependent constant (under the κ-semi-convexity assumptions). Wang [17,18] showed that the case $p = 2$ and $\delta > \kappa/2$ under our κ-semi-convexity assumptions implies a log-Sobolev inequality, which by the work of Bakry–Ledoux [2] (see also Ledoux [11]) implies the right (Gaussian) isoperimetric inequality. A more elementary approach was proposed by Bobkov [6], who considered the cases $p = 1, 2$ under our convexity assumptions on $(\mathbb{R}^n, |\cdot|, \mu)$. Bobkov's method is in fact very general; his results were generalized to all $p \geqslant 1$ by Barthe [3], and in fact the method may be carried over to treat general manifolds with density satisfying our convexity assumptions. Barthe and Kolesnikov [4] have obtained the most general results in this spirit (see in particular [4, Theorem 7.2]), treating general $p > 1$ under the convexity assumptions and $p \geqslant 2$ under the κ-semi-convexity ones, again, with dimension dependent constants in the conclusion.

The first dimension *independent* result was recently obtained in our previous work [14,13]. We showed that under the convexity assumptions, arbitrarily weak concentration (arbitrary slow α increasing to infinity) implies a linear isoperimetric inequality $\tilde{\mathcal{I}}(v) \geqslant c_\alpha v$, with c_α depending solely on α. In this respect, Theorem 1 complements our previous result, by showing that whenever α increases faster than linearly (corresponding to stronger-than-exponential concentration), one deduces a better-than-linear isoperimetric inequality.

The key ingredients in the proofs from [13] were diffusion semi-group estimates following Ledoux (see [11] and the references therein), and a result from Riemannian Geometry stating that under our convexity assumptions, $\mathcal{I}$ is necessarily a concave function on $(0, 1)$ (see [5] or [14] and the references therein). It has since become apparent that techniques from Riemannian Geometry and Geometric Measure Theory will play a fundamental role in any further progress in these directions. Indeed, the proofs of Theorems 1 and 2 are purely geometric, based on a generalized version of the Heintze–Karcher comparison theorem due to Bayle [5] and Morgan [16], and in fact no longer require any semi-group arguments. In this sense, we continue the geometric approach of Gromov [9], Buser [7] and Morgan [16] for deducing isoperimetric inequalities.

Theorems 1 and 2 have various further applications whenever our κ-semi-convexity assumptions are satisfied: obtaining isoperimetric inequalities on compact manifolds-with-density (generalizing [6, Corollary 2.2]), equivalence between transportation cost inequalities with different cost functions, and others. These will be described in detail in [12].

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