



Mathematical Problems in Mechanics

On the periodic orbits of the circular double Sitnikov problem

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Abstract

We introduce a restricted four body problem in a $2 + 2$ configuration extending the classical Sitnikov problem to the Double Sitnikov problem. The secondary bodies are moving on the same perpendicular line to the plane where the primaries evolve, so almost every solution is a collision orbit. We extend the solutions beyond collisions with a symplectic regularization and study the set of energy surfaces that contain periodic orbits. *To cite this article: H. Jiménez Pérez, E.A. Lacomba, C. R. Acad. Sci. Paris, Ser. I 347 (2009).*

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Résumé

Sur les orbites périodiques du problème de Sitnikov à quatre corps. On présente un problème à quatre corps en configuration $2 + 2$ qui étend le problème de Sitnikov au problème double de Sitnikov. Dans cette configuration presque toute solution du problème est une orbite avec collisions. On étend les solutions qui présentent des collisions en utilisant une régularisation symplectique et on étudie l'ensemble des surfaces d'énergie qui contiennent des orbites périodiques. *Pour citer cet article : H. Jiménez Pérez, E.A. Lacomba, C. R. Acad. Sci. Paris, Ser. I 347 (2009).*

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1. Introduction

We extend the classical Sitnikov problem [2] that is a special case of the restricted three body problem to a restricted four body problem in a $2 + 2$ configuration. This configuration consists in two massive bodies evolving on Keplerian orbits around their center of masses and two infinitesimal bodies evolving on the perpendicular line that cross the center of masses. The Double Sitnikov Problem consists in determining the evolution of the infinitesimal bodies under the Newtonian attraction of the massive bodies. Since the evolution of the secondaries is collinear we are interested in solutions with elastic bouncing at collisions. In this Note, we consider the circular restricted case of the double Sitnikov problem which is the integrable case. We study the periodic orbits on resonant tori.

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2. The circular double Sitnikov problem

We consider that primaries are evolving in circular Keplerian orbits and each one has mass $m_1 = m_2 = \frac{1}{2}$. The secondary masses are related by $\nu = c \mu$ with $c \in]0, 1]$. We choose the normalized masses $\alpha = \frac{1}{1+c}$ and $\beta = 1 - \alpha$. With these conditions the total energy of the system depends on α and μ as parameters as follows:

$$H = \frac{1}{2} \mathbf{p}^T M^{-1} \mathbf{p} - \frac{\alpha}{\sqrt{q_3^2 + 1/4}} - \frac{\beta}{\sqrt{q_4^2 + 1/4}} - \mu \frac{\beta}{q_3 - q_4}, \quad \text{where } M = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}. \quad (1)$$

The Hamiltonian vector field is obtained by $i_{X_H} \omega = dH$. The configuration space will be $\mathcal{P} = \{q_3 > q_4\}$ and the flow $\varphi(x, t) = \varphi_t(x)$ is not complete by the singularity due to collision. To avoid the singularity and to extend analytically the equations to the hyperplane $q_3 = q_4$ we perform a symplectic regularization with the mapping $\rho : T^*\mathbb{R}^2 \rightarrow T^*\mathbb{R}^2$ defined through the generating function of second type,

$$W(\mathbf{Q}, \mathbf{p}) = p_3 \left(Q_4 + \beta \frac{Q_1^2}{2} \right) + p_4 \left(Q_4 - \alpha \frac{Q_3^2}{2} \right).$$

Then the mapping $\rho : (\mathbf{Q}, \mathbf{P}) \mapsto (\mathbf{q}, \mathbf{p})$ is such that $\rho^*(\sum_i dp_i \wedge dq_i) = \sum_i dP_i \wedge dQ_i$ and, therefore $\rho \in Sp(T^*\mathbb{R}^2)$. Also, we consider the time rescaling $\frac{dt}{d\tau} = \alpha\beta Q_3^2$. The regularized Hamiltonian function is $L = \alpha\beta Q_3^2(H - h) \circ \rho$; this Hamiltonian function depends on α , h and μ as parameters and is valid only in the energy level $L = 0$ for each h fixed. If $\mathbf{z} = (Q_3, Q_4, P_3, P_4)$ we have explicitly:

$$L = \frac{1}{2} (\alpha\beta P_4^2 Q_3^2 + P_3^2) - 2\alpha\beta^2 \mu - \alpha\beta Q_3^2 \left[\frac{2\alpha}{\sqrt{(2Q_4 + \beta Q_3^2)^2 + 1}} + \frac{2\beta}{\sqrt{(2Q_4 - \alpha Q_3^2)^2 + 1}} + h \right].$$

Obtaining $\lim_{\mu \rightarrow 0} L_h(\mathbf{z}; \alpha, \mu) = L_h(\mathbf{z}; \alpha, 0)$ and reversing the process, since $\alpha\beta Q_3^2$ is not identically zero, we recover the Hamiltonian function without the term $\mu \frac{\beta}{q_3 - q_4}$. Finally, to extend the solutions in a continuous way beyond collisions (with elastic bouncing) it is necessary that the linear momentum be conserved and it is possible if and only if $c = 1$. We have the following:

Proposition 2.1. *In the circular double Sitnikov problem if $\mu = \nu$ then the flow $\varphi_t(x)$ of the limiting case $\mu \rightarrow 0$ can be extended to a complete flow in a natural way.*

Then the Hamiltonian system of the double Sitnikov problem is the triplet $(T^*\mathbb{R}^2, \omega, H)$ where

$$\omega = \sum_{i=3}^4 dp_i \wedge dq_i, \quad \text{and} \quad H = \frac{1}{2} |\mathbf{p}|^2 - \frac{1}{\sqrt{q_3^2 + 1/4}} - \frac{1}{\sqrt{q_4^2 + 1/4}}, \quad \mathbf{p} = (p_3, p_4).$$

3. Periodic orbits

Theorem 3.1. *The action-angle coordinates for the circular double Sitnikov problem takes the form:*

$$J(h_i) = \frac{\sqrt{2}}{\pi} (2E(k_i) - K(k_i) - \Pi(2k_i^2, k_i)), \quad \theta_i(t; h_i) = \frac{1}{\Omega_i} t(\nu, k) + \theta_{0,i}, \quad (2)$$

where $\Omega_i = \frac{\sqrt{2}}{4\pi(1-2k_i^2)} (2E(k_i) - K(k_i) + \Pi(2k_i^2, k_i))$ is the return time of the secondaries, $k_i = \frac{\sqrt{2+h_i}}{2}$ and $\theta_{0,i}$, for $i = 3, 4$, are constants determined by the initial conditions.

Theorem 3.2. *The solutions for the circular double Sitnikov problem can be written as*

$$(q_3, p_3, q_4, p_4)(t) = \left(\frac{k_3 s(\nu_3) d(\nu_3)}{1 - 2k_3^2 s^2(\nu_3)}, 2\sqrt{2}k_3 c(\nu_3), \frac{k_4 s(\nu_4) d(\nu_4)}{1 - 2k_4^2 s^2(\nu_4)}, 2\sqrt{2}k_4 c(\nu_4) \right),$$

where v_i are functions of t obtained inverting the function:

$$t = \int \frac{\sqrt{2}}{4(1 - 2k_i^2 \operatorname{sn}(v_i)^2)^2} dv_i, \tag{3}$$

and $s(v_i) \equiv \operatorname{sn}(v_i(t), k_i)$, $c(v_i) \equiv \operatorname{cn}(v_i(t), k_i)$, $d(v_i) \equiv \operatorname{dn}(v_i(t), k_i)$ are the sine, cosine, and delta amplitude Jacobi elliptic functions, and $k_i = \frac{\sqrt{2+h_i}}{2}$ for $i = 3, 4$.

It is possible to integrate the expression (3) with elliptic functions and elliptic integrals to obtain:

$$t = \frac{\sqrt{2}}{8(1 - 2k^2)} \left[2E(v) - v + \Pi(v, 2k^2) - 4k^2 \frac{\operatorname{sn}(v)\operatorname{cn}(v)\operatorname{dn}(v)}{1 - 2k^2\operatorname{sn}(v)^2} \right] + C, \tag{4}$$

where C is an arbitrary constant of integration. In [1] the reader will find a nice and complete study of this function.

Definition 3.3. We say that $\varphi(t)$ is a periodic solution of period τ with $\tau > 0$ if $\varphi(t + \tau) = \varphi(t)$ for all $t \in \mathbb{R}$ and there does not exist $\hat{\tau} \in (0, \tau)$ such that $\varphi(t + \hat{\tau}) = \varphi(t)$, i.e., τ is the minimum period.

We will use the notation $(p, q, n) = 1$ to mean that the greatest common divisor is $\operatorname{gcd}(p, q, n) = 1$, in other words, that the three numbers have not common factors at the same time.

Proposition 3.4. For every periodic solution of the double Sitnikov problem there exist 3-plets $(p, q, n) \in \mathbb{Z}^3$ such that $(p, q, n) = 1$, and $p > \frac{q}{2\sqrt{2}}$ and $p > \frac{n}{2\sqrt{2}}$ hold. The periods of these solutions are related to the partial energies by $\tau = 2p\pi = qT(h_1) = nT(h_2)$.

Definition 3.5. We say that an energy surface $\Sigma_{h_*} = H^{-1}(h_*)$ accepts a periodic solution if there exists $(p, q, n) \in \mathbb{N}^3$ with the properties:

- (P1) $(p, q, n) = 1$; and
- (P2) $p > \frac{q}{2\sqrt{2}}$, $p > \frac{n}{2\sqrt{2}}$ such that $h_* = T^{-1}(\frac{2\pi p}{q}) + T^{-1}(\frac{2\pi p}{n})$.

We denote the set of the constant energy surfaces that accept periodic orbits as

$$\mathfrak{M} = \{ \Sigma_h = H^{-1}(h) \mid h = h_* \}.$$

Theorem 3.6. In the circular double Sitnikov problem there exists a countable number of energy surfaces $\Sigma \in \mathfrak{M}$ that contains resonant tori foliated by periodic orbits. Moreover, the set of values $h_* \in H(T^*\mathbb{R}^2) \subset \mathbb{R}$ such that $\Sigma_{h_*} \in \mathfrak{M}$ is dense in $(-4, 0)$ and have zero measure in \mathbb{R} .

The proof of the theorem is an immediate consequence of the two following lemmas:

Lemma 3.7. For each $N \in \mathbb{N}$ the circular double Sitnikov problem has periodic solutions of period $2N\pi$.

We will just exhibit at least one periodic solution of period $\tau = 2N\pi$. This is immediate from the fact that there exists such periodic solutions in the circular (classical) Sitnikov problem.

Proof. For any $N \in \mathbb{N}$ we can choose the combination $p = N$ and $q = n = 1$ that produce $(p, q) = 1$ and $(p, n) = 1$ with $p > \frac{q}{2\sqrt{2}}$ and $p > \frac{n}{2\sqrt{2}}$ and Proposition 2.8 in [1] assures that there exists $h_1, h_2 \in (-2, 0)$ such that $T(h_1) = \frac{2\pi p}{q}$ and $T(h_2) = \frac{2\pi p}{n}$, then the hypersurface $H^{-1}(h_1 + h_2)$ contains a torus foliated by a family of periodic orbits with period $\tau = 2\pi N = T(h_1) = T(h_2)$. \square

Lemma 3.8. For each $N \in \mathbb{N}$ fixed, the circular double Sitnikov problem have a finite number of tori foliated by periodic orbits with period $\tau = 2N\pi$. The number,

$$8N\varphi(N) + \sum_{q < 2\sqrt{2}N} \varphi(q); \quad \text{with } \varphi(p) = p \prod_{n|p} \left(1 - \frac{1}{n}\right),$$

is an upper bound (although is not an optimum bound). $\varphi(p)$ is the totient function or Euler's phi function.

Proof. For each $p \in \mathbb{N}$ fixed there exist 3-plets $(p, q, n) \in \mathbb{N}^3$, where properties P1 and P2 of Definition 3.5 hold. Therefore, we search for the number C_p of 3-plets $(p, q, n) = 1$ coprimes. It is easy to see that for every $q < 2\sqrt{2}p$ and $(p, n) = 1$, the 3-plet (p, q, n) does not have common divisors. This triplets are exactly $(2\sqrt{2}p) \cdot (2\sqrt{2}\varphi(p)) = 8p\varphi(p)$.

Additionally, we must add all the couples (q, n) coprime such that (p, q) and (p, n) are not coprime. This means that for each integer $q < 2\sqrt{2}p$ with $(p, q) \neq 1$ we must add the number of coprimes $\varphi(q)$. Then we have:

$$C_p < 8p\varphi(p) + \sum_{q < 2\sqrt{2}p} \varphi(q). \quad (5)$$

Finally we must eliminate the elements that are in both sets, however the number (5) is an upper bound of the triplets $(p, q, n) \in \mathbb{N}^3$ where properties P1 and P2 holds.

The 3-plet $(p, q, n) \in \mathbb{N}^3$ induce a point $x = (2\pi \frac{p}{q}, 2\pi \frac{p}{n}) \in (T(h_3), T(h_4))$ such that the Lagrangian torus $\mathbb{T} = (\mu^{-1} \circ \mathcal{T}^{-1})(x)$ is foliated by periodic orbits of period $2N\pi$, therefore it is a resonant torus $\mathbb{T}_{\text{Res}} \subset T^*\mathbb{R}^2$. \square

Proof of Theorem 3.6. The first part of theorem is a consequence of the fact that the countable union of finite sets is a countable set. Using Lemmas 2 and 3 we have that the number of resonant tori are countable, and since each torus belongs to exactly one energy surface, the set \mathfrak{M} is countable too.

We define the map $\mathcal{T} : \mathfrak{g}^* \rightarrow \mathbb{R}^2$ by $(h_3, h_4) \mapsto (\frac{T(h_3)}{2\pi}, \frac{T(h_4)}{2\pi})$. For each rational point $y \in \text{Im } g(\mathcal{T})$ with $y = (\frac{r}{s}, \frac{u}{v})$, $(r, s) = 1$ and $(u, v) = 1$, we construct the point $(\frac{ru}{g}, \frac{su}{g}, \frac{rv}{g}) \in \mathbb{N}^3$ where $g = \text{gcd}(ru, su, rv)$. Since this point fulfills properties P1 and P2 of Definition 3.5, there exists a resonant torus foliated by periodic orbits with period $\tau = 2\frac{ru}{g}\pi = \frac{su}{g}T(h_3) = \frac{rv}{g}T(h_4)$. The set of rational values of \mathcal{T} defined by $RP := \text{Im } g(\mathcal{T}) \cap \mathbb{Q}^2$ is a dense subset of zero measure in $\text{Im } g(\mathcal{T})$. The mapping \mathcal{T} is continuous and then $\mathcal{T}^{-1}(RP) \subset \mathfrak{g}^*$ is a dense subset in the image of the momentum map μ . Now we construct the function $\mathcal{H} : \mathfrak{g}^* \rightarrow \mathbb{R}$ such that $x = (h_3, h_4) \mapsto h_3 + h_4$. It is immediate that $\mathcal{H}(\mathcal{T}^{-1}(RP)) \subset (-4, 0)$ is a dense subset by continuity, and have zero measure since RP is a countable set. \square

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