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Some relaxation results for functionals depending on constrained strain and chemical composition

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Abstract

We prove some relaxation results in the spirit of Anza Hafsa and Mandallena for integral functionals arising in the study of coherent thermochemical equilibria for multiphase solids. The energy density exhibits an explicit dependence on the deformation gradient and on a vector field representing the chemical composition. The deformation gradient satisfies a determinant type constraint and the chemical composition a constraint on the modulus. ***To cite this article: E. Zappale, H. Zorgati, C. R. Acad. Sci. Paris, Ser. I 347 (2009).***

Résumé

Résultats de relaxation pour des fonctionnelles dépendant d'une déformation avec contrainte et de la composition chimique. On prouve quelques résultats de relaxation dans le même esprit que Anza Hafsa et Mandallena pour des fonctionnelles intégrales provenant de l'étude de l'équilibre thermochimique pour les solides multiphases. La densité d'énergie considérée dépend du gradient de la déformation ainsi que d'un champ de vecteurs représentant la composition chimique du solide. Le gradient de déformation satisfait une contrainte sur son déterminant et la composition chimique une contrainte sur son module. ***Pour citer cet article : E. Zappale, H. Zorgati, C. R. Acad. Sci. Paris, Ser. I 347 (2009).***

Version française abrégée

Récemment les problèmes d'équilibre pour des énergies dépendant de deux champs indépendants ont fait l'objet de beaucoup d'attention. Ces modèles sont utiles pour la décomposition d'images, l'étude des matériaux micromagnétiques ainsi que les théories de Cosserat pour considérer les effets de flexion en élasticité non linéaire et l'analyse de l'équilibre thermochimique cohérent pour les solides multiphases. Dans [5] et [6], des résultats de relaxation pour de tels théories ont été obtenus. Notre but est d'étendre les résultats précédents pour les problèmes avec contrainte semblables à ceux considérés dans [1]. On considère donc l'énergie d'un solide occupant un domaine ouvert borné $\Omega \subset \mathbb{R}^N$, $N = 1, 2, 3$, ayant une frontière Lipschitzienne $\partial\Omega$. Le comportement du solide est caractérisé par une

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densité Borel mesurable $W:\mathbb{M}^{3\times N}\times\mathbb{R}^m\to[0,+\infty]$ dépendant de la déformation $u:\Omega\to\mathbb{R}^3$ du corps à travers son gradient $\nabla u\in\mathbb{M}^{3\times N}$, une matrice réelle $3\times N$, et d'un champ de vecteurs $m:\Omega\to\mathbb{R}^m$ représentant la composition chimique du solide. Soit g une fonction continue et affine par morceaux, l'énergie totale du solide $\mathcal{I}:W^{1,p}(\Omega;\mathbb{R}^3)\times L^q(\Omega;\mathbb{R}^m)\to[0,+\infty]$, $(p,q>1)$ est définie par

$$\mathcal{I}(u,m)=\begin{cases}\displaystyle\int_\Omega W(\nabla u,m)\,\mathrm{d}x & \text{si } u\equiv g \text{ sur } \partial\Omega,\\ +\infty & \text{sinon.}\end{cases}\tag{1}$$

En suivant la même approche que dans [1], on obtient le Théorème 1.2 qui est un résultat de relaxation sous le même type de contrainte sur le déterminant du gradient de la déformation en ajoutant une contrainte sur le module de la composition chimique. Ces conditions sont les suivantes :

– Pour $N=1$, il existe $\alpha,\alpha',\beta>0$ tel que pour tout $\xi\in\mathbb{M}^{3\times1}$, $v\in\mathbb{R}^m$,

$$\text{si } |\xi|\geqslant\alpha,\quad |v|\geqslant\alpha'\quad\text{alors}\quad W(\xi,v)\leqslant\beta\big(1+|\xi|^p+|v|^q\big),\ p,q>1.\tag{2}$$

– Pour $N=2$, il existe $\alpha,\alpha',\beta>0$ tel que pour tout $\xi=(\xi_1|\xi_2)\in\mathbb{M}^{3\times2}$, $v\in\mathbb{R}^m$,

$$\text{si } |\xi_1\wedge\xi_2|\geqslant\alpha,\quad |v|\geqslant\alpha'\quad\text{alors}\quad W(\xi,v)\leqslant\beta\big(1+|\xi|^p+|v|^q\big),\ p,q>1\tag{3}$$

où $\xi_1\wedge\xi_2$ représente le produit vectoriel des vecteurs colonnes ξ_1 et ξ_2.

– Pour $N=3$, il existe $\alpha',\beta>0$, et pour tout $\delta>0$, il existe $C_\delta>0$ tel que pour tout $\xi\in\mathbb{M}^{3\times3}$, $v\in\mathbb{R}^m$,

$$\text{si } |\det\xi|\geqslant\delta,\quad |v|\geqslant\alpha'\quad\text{alors}\quad W(\xi,v)\leqslant C_\delta\big(1+|\xi|^p\big)+\beta|v|^q,\ p,q>1.\tag{4}$$

– De plus, pour $N=3$, on suppose que W vérifie les hypothèses d'isotropie et d'indifférence matérielle suivantes,

$$W(P\xi Q,v)=W(\xi,v)\quad\text{pour tout } v\in\mathbb{R}^m,\ \xi\in\mathbb{M}^{3\times3}\text{ et } P,Q\in\mathrm{SO}(3),\tag{5}$$

où $\mathrm{SO}(3)=\{Q\in M^{3\times3}\colon\ Q^TQ=QQ^T=I \text{ et } \det Q=1\}$, avec Q^T la matrice transposée de Q et I la matrice identité.

1. Introduction

In this Note we give some relaxation results in the spirit of [1] for energies of the type (1) depending on the deformation of a solid and its chemical composition. The relaxation results take into account the constraints (2)–(4) which are determinant type constraint on the gradient of the deformation and a constraint on the modulus of the chemical composition. We emphasize that assumptions (2)–(4) have been made in order to take into account the fact that an infinite amount of energy is required to compress a finite line ($N=1$), surface ($N=2$) or volume ($N=3$) into zero line, surface or volume, with in addition a similar constraint on the concentration, in such a way as to be able to consider stored energy densities of the form

$$W(\xi,v):=|\xi|^p+|v|^q+k\big(|v|\big)+\begin{cases}h\big(|\xi|\big) & \text{if } N=1,\\ h\big(|\xi_1\wedge\xi_2|\big) & \text{if } N=2,\\ h\big(|\det\xi|\big) & \text{if } N=3\end{cases}$$

for all $\xi\in M^{3\times N}$, $v\in\mathbb{R}^m$, where $k:[0,+\infty[\,\to[0,+\infty]$, $h:[0,+\infty[\,\to[0,+\infty]$ are Borel measurable and such that for every $\delta>0$, there exists $r_\delta>0$ such that $h(t),k(\tau)\leqslant r_\delta$ for all $t,\tau\leqslant\delta$ (for example, $h(0)=+\infty$ and $h(t)=\frac{1}{t^s}$ if $t>0$ with $s>0$ and $k(0)=+\infty$, $k(\tau)=\frac{1}{\tau^\sigma}$, if $\tau>0$, with $\sigma>0$). We remark that the constraint on $|v|$ has been made in order to state the results with more generality, but the same results can be obtained considering v as an unconstrained field. Since our results generalize those in [1] to the case when the energy also depends on chemical composition, for the reader's convenience we will adopt the same notations as those in [1]. We also suppose for $N=1,2,3$ that W verifies the following coercivity condition: there exists $C>0$ such that

$$W(\xi,v)\geqslant C\big(|\xi|^p+|v|^q\big),\quad\forall(\xi,v)\in\mathbb{R}^{N\times3}\times\mathbb{R}^m.\tag{6}$$

We denote by $W_g^{1,p}(\Omega;\mathbb{R}^3)=\{\phi\in W^{1,p}(\Omega;\mathbb{R}^3),\ \phi=g \text{ on } \partial\Omega\}$. Define the relaxed energy $\mathcal{Q}^*\mathcal{I}: W^{1,p}(\Omega;\mathbb{R}^3)\times L^q(\Omega;\mathbb{R}^m)\to[0,+\infty]$, by

$$\mathcal{Q}^*\mathcal{I}(u,v)=\begin{cases}\int_\Omega Q^*W(\nabla u,v)\,\mathrm{d}x & \text{if } u\in W_g^{1,p}(\Omega;\mathbb{R}^3) \text{ and } v\in L^q(\Omega;\mathbb{R}^m),\\ +\infty & \text{otherwise,}\end{cases} \tag{7}$$

where Q^*W denotes the cross-quasiconvex envelope (see [5] and [6]) of W defined by

$$Q^*W(\xi,v)=\inf\left\{\int_D W(\xi+\nabla\theta,v+\eta)\,\mathrm{d}x\colon\ \theta\in W_0^{1,\infty}(D:\mathbb{R}^3),\ \eta\in L_0^\infty(D;\mathbb{R}^m)\right\},$$

where D is any cube in $\mathbb{R}^N$ and $L_0^\infty(D;\mathbb{R}^m)=\{\eta\in L^\infty(D;\mathbb{R}^m)$ such that $\int_D\eta\,\mathrm{d}x=0\}$. We define, in the same spirit as [1], following [4], the envelope

$$Z^*W(\xi,v)=\inf\left\{\int_{]0,1[^N} W(\xi+\nabla\theta,v+\eta)\,\mathrm{d}x\colon\ \theta\in \mathrm{Aff}_0(]0,1[^N;\mathbb{R}^3),\ \eta\in \mathrm{Cst}_0(]0,1[^N;\mathbb{R}^m)\right\}, \tag{8}$$

where $\mathrm{Aff}_0(D;\mathbb{R}^3)\subset W_0^{1,\infty}(D:\mathbb{R}^3)$ is the set of piecewise affine functions which vanish on the boundary of D and $\mathrm{Cst}_0(D;\mathbb{R}^m)\subset L_0^\infty(D;\mathbb{R}^m)$ is the set of piecewise constant, zero mean functions. These sets verify the following density results (see [3]), that will be crucial in the sequel:

Proposition 1.1. *Let $p,q>1$, then $\mathrm{Aff}_0(D;\mathbb{R}^3)$ is dense in $W_0^{1,p}(D;\mathbb{R}^3)$ with respect to the strong topology of $W^{1,p}(D;\mathbb{R}^3)$ and $\mathrm{Cst}_0(D;\mathbb{R}^m)$ is dense in $L_0^q(D;\mathbb{R}^m)$ with respect to the strong topology of $L^q(D;\mathbb{R}^m)$.*

The main result of this Note is the following theorem, generalizing Theorems 1.1, 1.2 and 1.3 in [1]:

Theorem 1.2. *Under the assumption* (6), *if* (2) *holds for the case $N=1$ (resp.* (3) *holds for the case $N=2$ and* (4), (5) *hold for the case $N=3$), then we have the following assertions*:

- $\inf\{\mathcal{I}(u,m)\colon\ u\in W_g^{1,p}(\Omega;\mathbb{R}^3),\ m\in L^q(\Omega;\mathbb{R}^m)\}=\inf\{\mathcal{Q}^*\mathcal{I}(u,m)\colon\ u\in W_g^{1,p}(\Omega;\mathbb{R}^3),\ m\in L^q(\Omega;\mathbb{R}^m)\}$.
- *If $\{(u_n,m_n)\}_{n\geqslant1}$ is a minimizing sequence for $\mathcal{I}$ in $W_g^{1,p}(\Omega;\mathbb{R}^3)\times L^q(\Omega;\mathbb{R}^m)$ such that $(u_n,m_n)\rightharpoonup(u,m)$ in $W^{1,p}(\Omega;\mathbb{R}^3)\times L^q(\Omega;\mathbb{R}^m)$, then (u,m) is a minimizer for $\mathcal{Q}^*\mathcal{I}$ in $W_g^{1,p}(\Omega;\mathbb{R}^3)\times L^q(\Omega;\mathbb{R}^m)$.*
- *If (u,m) is a minimizer for $\mathcal{Q}^*\mathcal{I}$ in $W_g^{1,p}(\Omega;\mathbb{R}^3)\times L^q(\Omega;\mathbb{R}^m)$, then there exists a minimizing sequence $\{(u_n,m_n)\}_{n\geqslant1}$ for $\mathcal{I}$ in $W_g^{1,p}(\Omega;\mathbb{R}^3)\times L^q(\Omega;\mathbb{R}^m)$ such that $(u_n,m_n)\rightharpoonup(u,m)$ in $W^{1,p}(\Omega;\mathbb{R}^3)\times L^q(\Omega;\mathbb{R}^m)$.*

1.1. Preliminary and intermediate results

The function Z^*, introduced in (8) in analogy with what has been proven in the purely elastic framework in [4], verifies the following properties:

Proposition 1.3. *We have*

- *For every bounded open set $D\subset\mathbb{R}^N$ with $|\partial D|=0$ and every $(\xi,v)\in\mathbb{M}^{3\times N}\times\mathbb{R}^m$ $Z^*W(\xi,v)=\inf\{\frac{1}{|D|}\int_D W(\xi+\nabla\theta,v+\eta)\,\mathrm{d}x\colon\ \theta\in\mathrm{Aff}_0(D;\mathbb{R}^3),\ \eta\in\mathrm{Cst}_0(D;\mathbb{R}^m)\}$.*
- *If Z^*W is finite (i.e. $Z^*W(\xi,v)<+\infty\ \forall(\xi,v)\in\mathbb{M}^{3\times N}\times\mathbb{R}^m$) then it is convex in the directions $(a\otimes\nu,b)$ with $a\in\mathbb{R}^3$, $\nu\in S^{N-1}$, $b\in\mathbb{R}^m$, i.e. for every $\xi_1,\xi_2\in\mathbb{M}^{3\times N}$ with $\mathrm{rank}(\xi_1-\xi_2)\leqslant1$, every $v_1,v_2\in\mathbb{R}^m$ and every $0\leqslant\lambda\leqslant1$, $Z^*W(\lambda(\xi_1,v_1)+(1-\lambda)(\xi_2,v_2))\leqslant\lambda Z^*W(\xi_1,v_1)+(1-\lambda)Z^*W(\xi_2,v_2)$.*
- *If Z^*W is finite then it is continuous.*
- *For every bounded open set $D\subset\mathbb{R}^N$ with $|\partial D|=0$, every $(\xi,v)\in\mathbb{M}^{3\times N}\times\mathbb{R}^m$, every $(\theta,\eta)\in\mathrm{Aff}_0(D;\mathbb{R}^3)\times\mathrm{Cst}_0(D;\mathbb{R}^m)$ $Z^*W(\xi,v)\leqslant\frac{1}{|D|}\int_D Z^*W(\xi+\nabla\theta,v+\eta)\,\mathrm{d}x$.*

In the sequel we will exploit the following result, (similar to the quasiconvex context (see [2])), whose proof can be found in [6] (see Theorem 4.16 therein), which states that if W is continuous and finite then $Q^*W = Z^*W$. Moreover, an argument analogous to that used in Proposition 1.9 of [1] guarantees the following equalities:

Proposition 1.4. *If Z^*W is finite then, $Q^*W = Q^*(Z^*W) = Z^*W$ and for $N = 1$ we have $Z^*W = W^{**}$, the lower semicontinuous convex envelope of W.*

Let $\mathcal{I}$ be defined as in (1), where W satisfies assumptions (2) if $N = 1$, (3) if $N = 2$ and (4) and (5) if $N = 3$ and is coercive in the sense of (6). Let $\mathcal{Q}^*\mathcal{I}$ be defined by (7) and let $\overline{\mathcal{I}}$ be the lower semicontinuous envelope of $\mathcal{I}$ with respect to the weak topology of $W^{1,p}(\Omega;\mathbb{R}^3) \times L^q(\Omega;\mathbb{R}^m)$. We recall the following result, in the unconstrained case, which can be found in [5] (see [6] for the case $p = q$):

Theorem 1.5. *Let $W : \mathbb{R}^{3\times N} \times \mathbb{R}^m \to \mathbb{R}$ be a continuous function of (p,q)-polynomial growth. Consider the functionals $\mathcal{F}$ defined as in (1), $\mathcal{Q}^*\mathcal{F}$ defined as in (7) and $\overline{\mathcal{F}}$ be the lower semicontinuous envelope of $\mathcal{F}$ with respect to the weak topology of $W^{1,p}(\Omega;\mathbb{R}^3) \times L^q(\Omega;\mathbb{R}^m)$. Then, $\overline{\mathcal{F}} = \mathcal{Q}^*\mathcal{F}$.*

A crucial step to prove Theorem 1.2 is the following theorem:

Theorem 1.6. *If Z^*W is of (p,q)-polynomial growth, i.e. $Z^*W(\xi,v) \leqslant c(1+|\xi|^p+|v|^q)$ for all $(\xi,v) \in \mathbb{M}^{3\times N} \times \mathbb{R}^m$ and some $c > 0$, then $\overline{\mathcal{I}} = \mathcal{Q}^*\mathcal{I}$.*

Indeed, the statement of Theorem 1.2 will be achieved, if under those assumptions, one can prove that Z^*W is of (p,q)-polynomial growth and apply Theorem 1.6. Precisely, the following proposition holds

Proposition 1.7. *If* (2) *holds for the case $N = 1$,* (*resp.* (3) *for the case $N = 2$ and* (4), (5) *for the case $N = 3$*) *then, Z^*W is of (p,q)-polynomial growth.*

In order to prove Theorem 1.6 we need the following result, whose proof has been omitted since it follows along the lines of the proof of Proposition 1.10 in [1] (see also Theorem 4.16 in [6]):

Proposition 1.8. *Let $\mathcal{J}_0^*, \mathcal{J}_1^* : W^{1,p}(\Omega;\mathbb{R}^3) \times L^q(\Omega;\mathbb{R}^m) \to [0,+\infty]$, be respectively defined by*

$$\mathcal{J}_0^* = \inf\Big\{\liminf \mathcal{I}(u_n,v_n) : (u_n,v_n) \in \mathrm{Aff}_g\big(\Omega;\mathbb{R}^3\big) \times \mathrm{Cst}\big(\Omega;\mathbb{R}^m\big),\\ (u_n,v_n) \rightharpoonup (u,v) \text{ in } W^{1,p}\big(\Omega;\mathbb{R}^3\big) \times L^q\big(\Omega;\mathbb{R}^m\big)\Big\}$$

and

$$\mathcal{J}_1^* = \inf\Big\{\liminf \mathcal{Z}^*\mathcal{I}(u_n,v_n) : (u_n,v_n) \in \mathrm{Aff}_g\big(\Omega;\mathbb{R}^3\big) \times \mathrm{Cst}\big(\Omega;\mathbb{R}^m\big),\\ (u_n,v_n) \rightharpoonup (u,v) \text{ in } W^{1,p}\big(\Omega;\mathbb{R}^3\big) \times L^q\big(\Omega;\mathbb{R}^m\big)\Big\},$$

where $\mathcal{Z}^*\mathcal{I}(u,v) = \int_\Omega Z^*W(\nabla u, v)\,\mathrm{d}x$. Then $\mathcal{J}_0^* = \mathcal{J}_1^*$.

Proof of Theorem 1.6. Theorem 1.6, in turn, follows from Proposition 1.4 and from Proposition 1.8 below, observing that if Z^*W is of (p,q)-polynomial growth, it is finite and hence continuous, by Proposition 1.3. Moreover, the density result stated in Proposition 1.1 and the continuity with respect to the strong topology, guarantee that $\mathcal{J}_1^*(u,v) = \inf\{\liminf Z^*W(u_n,v_n) : u_n \in W_g^{1,p}(\Omega;\mathbb{R}^N), v_n \in L^q(\Omega;\mathbb{R}^m) : (u_n,v_n) \rightharpoonup (u,n) \text{ in } W^{1,p} \times L^q\}$. Then, Proposition 1.4 ensures that $Q^*W = Q^*[Z^*W]$, hence $\mathcal{J}_1^* = \mathcal{Q}^*\mathcal{I}$, from Theorem 1.5. On the other hand, for every $(u,v) \in W^{1,p}(\Omega;\mathbb{R}^3) \times L^q(\Omega;\mathbb{R}^m)$ and for any $(u_n,v_n) \rightharpoonup (u,v) \in W_g^{1,p}(\Omega;\mathbb{R}^3) \times L^q(\Omega;\mathbb{R}^m)$, we have $Z^*\mathcal{I}(u_n,v_n) \leqslant \mathcal{I}(u_n,v_n)$ for any $n \geqslant 1$. Consequently $\mathcal{J}_1^*(u,v) \leqslant \liminf_n Z^*\mathcal{I}(u_n,v_n) \leqslant \liminf_n \mathcal{I}(u_n,v_n)$ and so $\mathcal{J}_1 \leqslant \overline{\mathcal{I}}$. But $\overline{\mathcal{I}} \leqslant \mathcal{J}_0^*$ and $\mathcal{J}_0^* = \mathcal{J}_1^*$ by Proposition 1.8, hence $\overline{\mathcal{I}} = \mathcal{J}_1^*$ and the theorem follows. □

We devote the next section to the proof of Proposition 1.7 mainly focusing on the differences with respect to the quasiconvex case.

2. Proof of Proposition 1.7

Case $N = 1$: Let $Q :=]0,1[$ and assume that (2) is satisfied.

- The case when $|\xi| \geqslant \alpha$ and $|v| \geqslant \alpha'$ is obvious, since Z^*W is of (p,q)-polynomial growth.
- When $|\xi| < \alpha$ and $|v| \geqslant \alpha'$, we consider a function $\phi \in \mathrm{Aff}_0(Q;\mathbb{R}^m)$ as in the proof of Proposition 1.6 in [1] which implies that $|\xi + \nabla\phi| \geqslant \alpha$ and thus $Z^*W(\xi,v) \leqslant \int_\Omega W(\xi+\nabla\phi, v) \leqslant \beta\int_\Omega(1+|\xi+\nabla\phi|^p+|v|^q) \leqslant \beta 2^{2p}\max\{1,\alpha^p\}(1+|\xi|^p+|v|^q)$.
- When $|\xi| \geqslant \alpha$, $|v| < \alpha'$, we consider a function $\psi \in \mathrm{Cst}_0(Q;\mathbb{R}^m)$ in such a way to have $|v+\psi| \geqslant \alpha'$ and $Z^*W(\xi,v) \leqslant \int_\Omega W(\xi, v+\psi) \leqslant \beta\int_\Omega(1+|\xi|^p+|v+\psi|^q) \leqslant \beta 2^{2q}\max\{1,\alpha'^q\}(1+|\xi|^p+|v|^q)$.
- When $|\xi| < \alpha$, $|v| < \alpha'$, we use test-functions ϕ and ψ as done above and we get the result.

Case $N = 2$: Let $Q :=]0,1[^2$ and assume that (3) is satisfied.

- The case when $|\xi_1 \wedge \xi_2| \geqslant \alpha$ and $|v| \geqslant \alpha'$ is obvious.
- When $|\xi_1 \wedge \xi_2| \geqslant \alpha$, $|v| < \alpha'$ we consider the function $\psi \in \mathrm{Cst}_0(Q;\mathbb{R}^m)$ which gives that $|v+\psi| \geqslant \alpha'$ and proceeding as in the case $N = 1$ we obtain the result.
- When $|\xi_1 \wedge \xi_2| < \alpha$, $|v| \geqslant \alpha'$ we proceed exactly as in the proof of Proposition 1.7 in [1].
- When $|\xi_1 \wedge \xi_2| < \alpha$, $|v| < \alpha'$ we have the following lemma:

Lemma 2.1. *When condition* (3) *holds, there exists* $\gamma > 0$ *such that, if* $|\xi_1+\xi_2|, |\xi_1-\xi_2| \geqslant \alpha$ *and* $|v| < \alpha$ *then* $Z^*W(\xi,v) \leqslant \gamma(1+|\xi|^p+|v|^q)$.

Proof. The proof of the lemma uses the same construction made in [1] with some modifications due to the presence of the vector v. Thus we consider the three following cases: (i) $|\xi_1 \wedge \xi_2| \neq 0$; (ii) $|\xi_1 \wedge \xi_2| = 0$ with $\xi_1 \neq 0$; (iii) $|\xi_1 \wedge \xi_2| = 0$ with $\xi_2 \neq 0$. We consider the domain $D = \{(x_1,x_2) \in \mathbb{R}^2\colon\ x_1 - 1 < x_2 < x_1 + 1 \text{ and } -x_1 - 1 < x_2 < 1 - x_1\}$ and we define a function $\phi_t \in \mathrm{Aff}_0(D;\mathbb{R})$ by

$$\phi_t(x_1,x_2) = \begin{cases} -tx_1 + t(x_2+1) & \text{if } (x_1,x_2) \in \Delta_1 = \{(x_1,x_2) \in D\colon\ x_1 \geqslant 0, x_2 \leqslant 0\},\\ t(1-x_1) - tx_2 & \text{if } (x_1,x_2) \in \Delta_2 = \{(x_1,x_2) \in D\colon\ x_1 \geqslant 0, x_2 \geqslant 0\},\\ tx_1 + t(1-x_2) & \text{if } (x_1,x_2) \in \Delta_3 = \{(x_1,x_2) \in D\colon\ x_1 \leqslant 0, x_2 \geqslant 0\},\\ t(x_1+1) + tx_2 & \text{if } (x_1,x_2) \in \Delta_4 = \{(x_1,x_2) \in D\colon\ x_1 \leqslant 0, x_2 \leqslant 0\}. \end{cases}$$

Let $\Phi \in \mathrm{Aff}_0(D;\mathbb{R}^3)$ defined by $\Phi = (\phi_{\nu_1}, \phi_{\nu_2}, \phi_{\nu_3})$ with $\nu = \frac{\xi_1 \wedge \xi_2}{|\xi_1 \wedge \xi_2|}$ if (i) is satisfied, $|\nu| = 1$, $\langle \xi_1, \nu\rangle = 0$ if (ii) is satisfied and $|\nu| = 1$, $\langle \xi_2, \nu\rangle = 0$ if (iii) is satisfied, and $\nu = (\nu_1,\nu_2,\nu_3)$. Let $\psi \in \mathrm{Cst}_0(D;\mathbb{R}^m)$ defined by

$$\psi(x) = -a \quad \text{in } \Delta_1 \cup \Delta_3 \quad \text{and} \quad \psi(x) = a \quad \text{in } \Delta_2 \cup \Delta_4 \tag{9}$$

with $|a| = 2\alpha'$. Then we compute $(\xi + \nabla\Phi, v + \psi)$ and the proof follows as in [1], estimating $Z^*W(\xi,v)$ using its definition and the test function $(\xi + \nabla\Phi, v+\psi)$. $\square$

An argument entirely similar to that appearing in the proof of Proposition 1.7 of [1] for the case $N = 2$ works using the same function defined in (9) and applying the above lemma.

Case $N = 3$: We assume that (4) and (5) are verified and let $Q :=]0,1[^3$. The third result stated in Proposition 1.7 is achieved by a series of intermediate results. First we can observe, as in Lemma 4.4 in [1], that our subsequent analysis can be reduced to considering diagonal matrices ξ, since we have the following lemma:

Lemma 2.2. *If* (5) *holds, then* $Z^*W(P\xi Q, v) = Z^*W(\xi, v)$ *for all* $\xi \in \mathbb{R}^{3\times 3}$, $v \in \mathbb{R}^m$, *and all* $P, Q \in \mathrm{SO}(3)$.

Consequently, Proposition 1.7 will be proved observing, in analogy with [1], first that for every $\xi \in \mathbb{R}^{3\times 3}$, with $\det\xi \neq 0$, there exists $\zeta \in \mathbb{R}^{3\times 3}$ diagonal, and $P, Q \in \mathrm{SO}(3)$ such that $\xi = PQ^T\zeta Q$, $|\zeta| = |\xi|$ and by virtue of the previous lemma $Z^*W(\xi,v) = Z^*W(\zeta,v)$. Thus, it is enough to prove that Z^*W has (p,q)-polynomial growth on such ζ and all v, and to exploit the continuity of Z^*W and the density of the matrices of $\mathbb{R}^{3\times 3}$ with nonzero determinant, in the space of all 3×3 matrices. Hence, all we have to prove is that under assumption (4) and for every $\xi \in \mathbb{R}^{3\times3}$ diagonal, and $v \in \mathbb{R}^m$, $Z^*W(\xi, v)$ has (p,q)-polynomial growth. Firstly we begin with the following lemma:

Lemma 2.3. *Under assumption* (4) Z^*W *is finite.*

Proof. Observe that if ξ is such that $\det\xi \neq 0$ and v is such that $|v| \geqslant \alpha'$, Z^*W is finite. If $\det\xi \neq 0$ and $|v| \leqslant \alpha'$, we can proceed as in the cases $N = 1$ and $N = 2$, defining a test function $\psi \in \mathrm{Cst}_0(Q; \mathbb{R}^m)$ such that

$$\psi(x) = \begin{cases} \nu & \text{in }]0,1[\times]0,1[\times \left]0, \frac{1}{2}\right[, \\ -\nu & \text{in }]0,1[\times]0,1[\times \left]\frac{1}{2}, 1\right[\end{cases} \tag{10}$$

with $|\nu| = 2\alpha'$. If $\det\xi = 0$ and $|v| \geqslant \alpha'$, the proof is exactly analogous to that of Lemma 4.2 in [1]. It remains to consider the case when $\operatorname{rank}\xi = 0$ or 1 or 2 and $|v| < \alpha'$. As in [1] we first consider the case when $\operatorname{rank}\xi = 2$ and split the domain of definition of Z^*W in eight subdomains, suitably chosen and repeat the same construction adopted therein for a test function $\phi(x)$ and define a test function $\psi(x)$ alternatively equal to $-a$ and a, with $|a| = 2\alpha'$ in each subdomain in such a way to repeat the same argument of the cases $N = 1$ and $N = 2$ to estimate the norm of the function $v + \psi$ and the determinant of $\xi + \nabla\phi(x)$ and obtaining the right estimate for $Z^*W(\xi, v)$. The cases $\operatorname{rank}\xi = 1$ with $|v| < \alpha'$ can be reduced to the step $\operatorname{rank}\xi = 2$ and finally the case $\xi = 0$ to the case $\operatorname{rank}\xi = 1$. □

Finally we have to prove the following proposition:

Proposition 2.4. *Under assumption* (4) *and for every* $\xi \in \mathbb{R}^{3\times3}$ *diagonal and for every* $v \in \mathbb{R}^m$, Z^*W *has* (p,q)-*polynomial growth.*

Proof. By Lemma 2.3 Z^*W is finite. Moreover, for every $(\xi, v) \in \mathbb{R}^{3\times3} \times \mathbb{R}^m$ verifying $|\xi|^2 \leqslant 3$ and $|v| \leqslant \alpha'$, there exists $C_0 > 0$ such that $Z^*W(\xi, v) \leqslant C_0$. On the other hand

- Assumption (4) ensures that Z^*W has (p,q)-polynomial growth if $|\det\xi| \geqslant 1$ and $|v| \geqslant \alpha'$.
- The same argument used in the proof of Lemma 4.3 in [1] ensures that Z^*W has polynomial growth when $|\det\xi| \leqslant 1$ and $|v| \geqslant \alpha'$, using the fact that assumption (4) provides a bound from above of the type $\beta|v|^q$.
- When $|\det\xi| \geqslant 1$ and $|v| < \alpha'$, we consider the function $\psi \in \mathrm{Cst}_0(Q; \mathbb{R}^3)$ defined in (10) and we proceed as in the cases $N = 1$ and $N = 2$ we get the result, taking into account that (4) provides a bound from above of the type $C_1(1 + |\xi|^p)$.
- When $|\det\xi| \leqslant 1$, $|\xi|^2 \geqslant 3$ and $|v| < \alpha'$ we adapt the proof of [1] considering the presence of v and using a function ψ analogous to (9) alternating the values a and $-a$ with $|a| = 2\alpha'$ in the decomposition of the domain used in [1]. □

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