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Note on Diophantine inequality and Linear Artin Approximation over a local ring

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Abstract

G. Rond [G. Rond, Approximation diophantienne dans les corps de séries en plusieurs variables, Ann. Institut Fourier 56 (2) (2006) 299–308. [10]] has proved Linear version of Artin Approximation theorem (LAA) and Diophantine inequality for a single homogeneous polynomial equation in two unknowns with coefficients in a formal (or convergent) power series ring over a field. M. Hickel and H. Ito, S. Izumi have generalized Rond's result to certain good local domains, independently, in 2008. This is a complementary Note to theirs. The most important point is that we can delete the equicharacteristic assumption in both papers. ***To cite this article: M. Hickel et al., C. R. Acad. Sci. Paris, Ser. I 347 (2009).***

Résumé

Inégalités Diophantiennes et théorème d'Approximation d'Artin Linéaire pour un anneau local. G. Rond [G. Rond, Approximation diophantienne dans les corps de séries en plusieurs variables, Ann. Institut Fourier 56 (2) (2006) 299–308. [10]] a démontré une version Linéaire du théorème d'Approximation d'Artin et des inégalités Diophantiennes pour un polynôme homogène en deux variables à coefficients dans un anneau de séries formelles (ou convergentes) sur un corps. M. Hickel et H. Ito, S. Izumi ont généralisé (indépendamment, en 2008) les résultats de G. Rond à certains anneaux locaux convenables. L'objectif principal de cette Note, complémentaire à ces deux articles, est d'indiquer que l'hypothèse d'égale caractéristique utilisée dans les deux articles est inutile. ***Pour citer cet article : M. Hickel et al., C. R. Acad. Sci. Paris, Ser. I 347 (2009).***

1. The paper [2] puts emphasis on the deduction of Linear version of Artin Approximation theorem (LAA). Diophantine inequality for Rees valuations appears in the proof of Theorem 1.1 in [2]. On the other hand [5] puts emphasis on the proof of Diophantine inequality for $\mathfrak{m}$-valuations. The notion of Rees valuations is a little restrictive in comparison to that of $\mathfrak{m}$-valuations (see [8], [9] Appendix) but it is sufficient for the proof of Theorem 1.1 in [2].

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2. The LAA of [2] is stronger than LAA of [5] in the point that [2] treats a finite number of homogeneous polynomials, a simultaneous equations, whereas [5] treat only a single equation. Furthermore, the last two authors have assumed that the local domain is analytically irreducible in [5]. This is superfluous (and not assumed in [2]). This is because of the important theorem of D. Popescu and others (cf. [7]) that an excellent Henselian local ring has the Artin approximation property. Applying this property to the simple equation $XY = 0$, we see that an excellent and Henselian local domain is analytically irreducible.

3. There is no need to assume equicharacteristic property of the local domain A in both papers. Let $(A, \mathfrak{m})$ be an excellent Henselian domain and $\boldsymbol{v}$ be a Rees valuation of $\mathfrak{m}$ (i.e. an $\mathfrak{m}$-valuation associated with $\mathfrak{m}$), $(V_{\boldsymbol{v}}, \mathfrak{m}_{\boldsymbol{v}})$ its valuation ring and $(\hat{V}_{\boldsymbol{v}}, \hat{\mathfrak{m}}_{\boldsymbol{v}})$ the completion of $V_{\boldsymbol{v}}$ with respect to $\boldsymbol{v}$.

In [2], equicharacteristic property is used (p. 755) to show that $\hat{V}_{\boldsymbol{v}}$ $(= \hat{S}_1$ in [2]) is isomorphic to the formal power series ring over $\hat{V}_{\boldsymbol{v}}/\hat{\mathfrak{m}}_{\boldsymbol{v}}$ (Cohen's structure theorem for equicharacteristic complete local rings) and then Greenberg's LAA [1] is applied to polynomial equations with coefficients in $\hat{V}_{\boldsymbol{v}}$. We can, however, apply Greenberg's theorem without the (superfluous) equicharacteristic assumption. The reason is that Greenberg's theorem requires only that the DVR ring is Henselian and that the field of quotients of its completion is a separable extention of its field of quotients. Our $\hat{V}_{\boldsymbol{v}}$ is obviously Henselian as any complete Noetherian local ring, see [6]. The separability condition is trivial since the completion of $\hat{V}_{\boldsymbol{v}}$ is itself.

In [5], the authors have adopted the equicharacteristic assumption to show that $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_p$ are $\mathfrak{m}$-valuations. This follows, however, from [4] (10.4.3), without such an assumption.

4. In [2], Sections 2, 3, 4, 5, the author uses implicitly the fact, [3] Lemma 1.1, that each Rees valuation of $\mathfrak{m}$ extends uniquely as a Rees valuation of $\mathfrak{m}.\hat{A}$ to apply Rees version of Izumi's theorem which is stated in the case of a complete domain in [9] (E) p. 409. In the characterizing properties (c) of $\mathfrak{m}$-valuations at the beginning of [5], Section 2, the authors missed to add the condition that $k_{\boldsymbol{v}}$ is finitely generated over k.

5. Thus we have the following improved results, deleting the equicharacteristic assumption:

Theorem A *(Diophantine inequality: cf. [2], (••); [5], (3.1)). Let $(A, \mathfrak{m})$ be an excellent Henselian local domain and let $K := Q(A)$ denote its field of quotients and let $\boldsymbol{v} : K \to \overline{\mathbb{R}}$ be an $\mathfrak{m}$-valuation. If $z \in \hat{K} \setminus K$ is algebraic over K, then we have*

$$\exists a > 0 \quad \exists c > 0 \quad \forall x \in A \quad \forall y \in A^*\colon \left| z - \frac{x}{y} \right|_{\hat{\boldsymbol{v}}} > c|y|_{\boldsymbol{v}}^a.$$

Theorem B *(LAA: cf. [2], (1.1)). Let $(A, \mathfrak{m})$ be an excellent Henselian local domain and let I be a homogeneous ideal in $A[X, Y]$. Then the Artin function of I is bounded by an affine function. That is, denoting by $\nu_{\mathfrak{m}}()$ the order with respect to the maximal ideal, there exists two constants $\alpha, \beta \in \mathbb{R}_{\geqslant 0}$ such that the following holds.*

$$\forall (x,\ y) \in A^2\colon \quad \left\{ \begin{array}{c} \forall P \in I\colon \quad \nu_{\mathfrak{m}}(P(x, y)) > \alpha i + \beta \\ \Longrightarrow \\ \exists (\bar{x}, \bar{y}) \in A^2\colon \quad \forall P \in I,\ P(\bar{x}, \bar{y}) = 0 \ \wedge\ \nu_{\mathfrak{m}}(\bar{x} - x) > i,\ \nu_{\mathfrak{m}}(\bar{y} - y) > i. \end{array} \right\}.$$

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