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A generalization of the classical Cesàro–Volterra path integral formula

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Abstract

If a symmetric matrix field $\boldsymbol{e}$ of order three satisfies the Saint Venant compatibility conditions in a simply-connected domain Ω in $\mathbb{R}^3$, there then exists a displacement field $\boldsymbol{u}$ of Ω such that $\boldsymbol{e} = \frac{1}{2}(\nabla \boldsymbol{u}^T + \nabla \boldsymbol{u})$ in Ω. If the field $\boldsymbol{e}$ is sufficiently smooth, the displacement $\boldsymbol{u}(x)$ at any point $x \in \Omega$ can be explicitly computed as a function of $\boldsymbol{e}$ and **CURL** $\boldsymbol{e}$ by means of a Cesàro–Volterra path integral formula inside Ω with endpoint x.

We assume here that the components of the field $\boldsymbol{e}$ are only in $L^2(\Omega)$, in which case the classical path integral formula of Cesàro and Volterra becomes meaningless. We then establish the existence of a "Cesàro–Volterra formula with little regularity", which again provides an explicit solution $\boldsymbol{u}$ to the equation $\boldsymbol{e} = \frac{1}{2}(\nabla \boldsymbol{u}^T + \nabla \boldsymbol{u})$ in this case. ***To cite this article: P.G. Ciarlet et al., C. R. Acad. Sci. Paris, Ser. I 347 (2009).***

Résumé

Une généralisation de la formule classique de l'intégrale curviligne de Cesàro–Volterra. Si un champ $\boldsymbol{e}$ de matrices symétriques d'ordre trois vérifie les conditions de compatibilité de Saint Venant dans un domaine simplement connexe Ω de $\mathbb{R}^3$, alors il existe un champ $\boldsymbol{u}$ de déplacements de Ω tel que $\boldsymbol{e} = \frac{1}{2}(\nabla \boldsymbol{u}^T + \nabla \boldsymbol{u})$ dans Ω. Si le champ $\boldsymbol{e}$ est suffisamment régulier, le déplacement $\boldsymbol{u}(x)$ peut être calculé explicitement en tout point $x \in \Omega$ comme une fonction de $\boldsymbol{e}$ et de **CURL** $\boldsymbol{e}$, au moyen d'une intégrale curviligne de Cesàro–Volterra le long d'un chemin contenu dans Ω et d'extrémité x.

On suppose ici que les composantes du champ $\boldsymbol{e}$ sont seulement dans $L^2(\Omega)$, auquel cas la formule intégrale de Cesàro–Volterra n'a pas de sens. On établit alors l'existence d'une « formule de Cesàro–Volterra avec peu de régularité », qui donne à nouveau dans ce cas une solution explicite $\boldsymbol{u}$ de l'équation $\boldsymbol{e} = \frac{1}{2}(\nabla \boldsymbol{u}^T + \nabla \boldsymbol{u})$. ***Pour citer cet article : P.G. Ciarlet et al., C. R. Acad. Sci. Paris, Ser. I 347 (2009).***

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1. Introduction

Latin indices range in the set $\{1, 2, \dots, n\}$ for some $n \geqslant 2$ and the summation convention with respect to repeated Latin indices is used in conjunction with this rule. The sets of all real matrices of order n, of all real symmetric matrices of order n, and of all real antisymmetric matrices of order n, are respectively denoted $\mathbb{M}^n$, $\mathbb{S}^n$, and $\mathbb{A}^n$.

It is well known that, if Ω is a *simply-connected* open subset of $\mathbb{R}^n$ and if a matrix field $\boldsymbol{e} = (e_{ij}) \in \mathcal{C}^2(\Omega; \mathbb{S}^n)$ satisfies the *Saint Venant compatibility conditions*

$$\partial_{lj} e_{ik} + \partial_{ki} e_{jl} - \partial_{li} e_{jk} - \partial_{kj} e_{il} = 0 \quad \text{in } \mathcal{C}^0(\Omega), \tag{1}$$

then there exists a vector field $\boldsymbol{u} = (u_i) \in \mathcal{C}^3(\Omega; \mathbb{R}^n)$ that satisfies the equations

$$\frac{1}{2}(\partial_j u_i + \partial_i u_j) = e_{ij} \quad \text{in } \Omega. \tag{2}$$

Besides, all other solutions $\widetilde{\boldsymbol{u}} = (\widetilde{u}_i) \in \mathcal{C}^3(\Omega; \mathbb{R}^n)$ to the equations $\frac{1}{2}(\partial_j \widetilde{u}_i + \partial_i \widetilde{u}_j) = e_{ij}$ in Ω are of the form

$$\widetilde{\boldsymbol{u}}(x) = \boldsymbol{u}(x) + \boldsymbol{a} + \boldsymbol{A}\boldsymbol{ox}, \quad x \in \Omega, \quad \text{for some } \boldsymbol{a} \in \mathbb{R}^n \text{ and } \boldsymbol{A} \in \mathbb{A}^n. \tag{3}$$

It is less known (Gurtin [13] constitutes an exception) that an *explicit solution* $\boldsymbol{u} = (u_i)$ *to the equations* (2) can be given in the form of the following *Cesàro–Volterra path integral formula*, so named after Cesàro [4] and Volterra [15]: Let $\gamma(x)$ be any path of class $\mathcal{C}^1$ contained in Ω and joining a point $x_0 \in \Omega$ (considered as fixed) to any point $x \in \Omega$. Then

$$u_i(x) = \int_{\gamma(x)} \left\{ e_{ij}(y) + \big(\partial_k e_{ij}(y) - \partial_i e_{kj}(y)\big)(x_k - y_k) \right\} \mathrm{d}y_j, \quad x \in \Omega. \tag{4}$$

It can then be verified that each value $u_i(x)$ computed by formula (4) is independent of the path chosen for joining x_0 to x, thanks to the compatibility conditions (1).

If $n = 3$, the Cesàro–Volterra path integral formula (4) can be equivalently rewritten in *vector-matrix form*, as

$$\boldsymbol{u}(x) = \int_{\gamma(x)} \boldsymbol{e}(y)\,\mathbf{d}\boldsymbol{y} + \int_{\gamma(x)} \boldsymbol{yx} \wedge \big(\big[\mathbf{CURL}\,\boldsymbol{e}(y)\big]\,\mathbf{d}\boldsymbol{y}\big), \quad x \in \Omega, \tag{5}$$

where $\wedge$ designates the vector product in $\mathbb{R}^3$, and **CURL** designates the matrix curl operator.

The sufficiency of the Saint Venant compatibility conditions (1) was recently shown to hold under substantially *weaker regularity assumptions on the given tensor field* $\boldsymbol{e} = (e_{ij})$, according to the following result, due to Ciarlet and Ciarlet, Jr. [5]: *Let Ω be a bounded and simply-connected open subset of $\mathbb{R}^n$ with a Lipschitz-continuous boundary, and let there be given functions $e_{ij} = e_{ji} \in L^2(\Omega)$ that satisfy the "Saint Venant compatibility conditions with little regularity", viz.,*

$$\partial_{lj} e_{ik} + \partial_{ki} e_{jl} - \partial_{li} e_{jk} - \partial_{kj} e_{il} = 0 \quad \text{in } H^{-2}(\Omega). \tag{6}$$

Then there exists a vector field $(u_i) \in H^1(\Omega; \mathbb{R}^3)$ that satisfies

$$\frac{1}{2}(\partial_j u_i + \partial_i u_j) = e_{ij} \quad \text{in } L^2(\Omega). \tag{7}$$

Besides, all the other solutions $\widetilde{\boldsymbol{u}} = (\widetilde{u}_i) \in H^1(\Omega; \mathbb{R}^3)$ to the equations $\frac{1}{2}(\partial_j \widetilde{u}_i + \partial_i \widetilde{u}_j) = e_{ij}$ are again of the form (3) (this result has since then been extended in various ways; see Geymonat and Krasucki [9], Ciarlet, Ciarlet, Jr., Geymonat and Krasucki [6], and Amrouche, Ciarlet, Gratie and Kesavan [2]).

Clearly, the "classical" Cesàro–Volterra path integral formula (4) becomes meaningless when the functions e_{ij} satisfying (6) are only in the space $L^2(\Omega)$. The question then naturally arises as to whether there exists any "*Cesàro–Volterra formula with little regularity*", which (i) would again provide an explicit solution to the equations (7) when the functions e_{ij} are only in $L^2(\Omega)$ and (ii) would in some way resemble (4).

The purpose of this Note is to provide a positive answer to this question. Complete proofs will be found in [8].

2. A Poincaré lemma with little regularity

A *domain* in $\mathbb{R}^n$ is an open, bounded, connected subset of $\mathbb{R}^n$, with a Lipschitz-continuous boundary. The mapping $\boldsymbol{T} = (T_i)$ defined in the next theorem (for a proof, see Corollaries 2.3 and 2.4, Chapter 1, of Girault and Raviart [12]; or Theorem 2' of Bourgain and Brezis [3] for the extension to L^p spaces, $1 < p < \infty$) plays a key role in our approach.

Theorem 1. *Let Ω be a domain in $\mathbb{R}^n$. Then there exists a linear and continuous operator*

$$\boldsymbol{T} = (T_i): L_0^2(\Omega) := \left\{ v \in L^2(\Omega); \int_\Omega v \, \mathrm{d}x = 0 \right\} \to H_0^1(\Omega; \mathbb{R}^n), \tag{8}$$

such that

$$-\operatorname{div}(\boldsymbol{T} v) = v \quad \textit{for all } v \in L_0^2(\Omega). \tag{9}$$

Our approach for finding a Cesàro–Volterra formula with little regularity also relies on the following *Poincaré lemma with little regularity*, due to Ciarlet and Ciarlet, Jr. [5] (for recent extensions of this result, see Amrouche, Ciarlet and Ciarlet, Jr. [1], Geymonat and Krasucki [10,11] and, especially, S. Mardare [14]).

Theorem 2. *Let Ω be a simply-connected domain in $\mathbb{R}^n$, and let $f_i \in H^{-1}(\Omega)$ be distributions that satisfy $\partial_i f_j - \partial_j f_i = 0$ in $H^{-2}(\Omega)$. Then there exists a function $u \in L^2(\Omega)$, unique up to an additive constant, such that $\partial_i u = f_i$ in $H^{-1}(\Omega)$.*

We first show that, even under the weak regularity assumptions of Theorem 1, there is a way to "compute" a solution $u \in L^2(\Omega)$ to the equations $\partial_i u = f_i$ in $H^{-1}(\Omega)$.

In what follows, $\langle \cdot, \rangle$ denotes the duality pairing between a topological space and its dual space.

Theorem 3. *Let Ω be a simply-connected domain in $\mathbb{R}^n$, let the space $\mathcal{D}_0(\Omega)$ be defined as*

$$\mathcal{D}_0(\Omega) := \left\{ \varphi \in \mathcal{D}(\Omega); \int_\Omega \varphi \, \mathrm{d}x = 0 \right\}, \tag{10}$$

and let $f_i \in H^{-1}(\Omega)$ be distributions that satisfy $\partial_i f_j - \partial_j f_i = 0$ in $H^{-2}(\Omega)$. Then a function $u \in L^2(\Omega)$ satisfies $\partial_i u = f_i$ in $H^{-1}(\Omega)$ if and only if

$$\langle u, \varphi \rangle = \langle f_i, T_i \varphi \rangle \quad \textit{for all } \varphi \in \mathcal{D}_0(\Omega), \tag{11}$$

where $\boldsymbol{T} = (T_i): L_0^2(\Omega) \to H_0^1(\Omega; \mathbb{R}^n)$ is the continuous linear operator defined in Theorem 1.

Interestingly, the solution to the equations $\partial_i u = f_i$ in $H^{-1}(\Omega)$ can also be found by solving a *variational problem* (cf. (12) below), which clearly satisfies all the assumptions of the *Lax–Milgram lemma*:

Theorem 4. *Let Ω be a simply-connected domain in $\mathbb{R}^n$, let the space $L_0^2(\Omega)$ be defined as in* (8)*, and let there be given distributions $f_i \in H^{-1}(\Omega)$ that satisfy $\partial_i f_j - \partial_j f_i = 0$ in $H^{-2}(\Omega)$.*

Then the variational problem: Find a function $u \in L_0^2(\Omega)$ such that

$$\langle u, v \rangle = \langle f_i, T_i v \rangle \quad \textit{for all } v \in L_0^2(\Omega), \tag{12}$$

has a unique solution, which is also a solution to the equations $\partial_i u = f_i$ in $H^{-1}(\Omega)$.

3. A Cesàro–Volterra formula with little regularity

Given functions $e_{ij} = e_{ji} \in L^2(\Omega)$ that satisfy the compatibility conditions (6), the classical Cesàro–Volterra path integral formula (4) becomes meaningless. But we nevertheless show that there is still a way in this case to "compute" a solution $\boldsymbol{u} = (u_i) \in H^1(\Omega; \mathbb{R}^n)$ to Eqs. (7).

This objective is achieved by means of an *explicit expression* in terms of the data $e_{ij} \in L^2(\Omega)$ of the duality pairings $\langle \boldsymbol{u}, \boldsymbol{\varphi} \rangle := \langle u_i, \varphi_i \rangle = \int_\Omega u_i \varphi_i \, dx$ for all vector fields $\boldsymbol{\varphi} = (\varphi_i) \in \mathcal{D}(\Omega; \mathbb{R}^n)$ that satisfy $\int_\Omega \varphi_i \, dx = \int_\Omega (x_j \varphi_i - x_i \varphi_j) \, dx = 0$. By reference with the classical Cesàro–Volterra path integral formula, we will say that relations (14) constitute the *Cesàro–Volterra formula with little regularity* (this terminology will be further substantiated in Theorem 7).

Theorem 5. *Let Ω be a simply-connected domain in $\mathbb{R}^n$, let the space $\mathcal{D}_1(\Omega; \mathbb{R}^n)$ be defined as*

$$\mathcal{D}_1(\Omega; \mathbb{R}^n) := \left\{ \boldsymbol{\varphi} = (\varphi_i) \in \mathcal{D}(\Omega; \mathbb{R}^n); \int_\Omega \varphi_i \, dx = \int_\Omega (x_j \varphi_i - x_i \varphi_j) \, dx = 0 \right\}, \tag{13}$$

and let there be given a matrix field $\boldsymbol{e} = (e_{ij}) \in L^2(\Omega; \mathbb{S}^3)$ whose components $e_{ij} = e_{ji} \in L^2(\Omega)$ satisfy the Saint Venant compatibility conditions with little regularity (6)*.*

Then a vector field $\boldsymbol{u} = (u_i) \in H^1(\Omega; \mathbb{R}^n)$ satisfies Eqs. (7) *if and only if*

$$\langle u_i, \varphi_i \rangle = \langle e_{ij}, T_i \varphi_j + \partial_k [T_i (T_j \varphi_k - T_k \varphi_j)] \rangle \quad \text{for all } \boldsymbol{\varphi} = (\varphi_i) \in \mathcal{D}_1(\Omega; \mathbb{R}^n), \tag{14}$$

where $\boldsymbol{T} = (T_i) : L_0^2(\Omega) \to H_0^1(\Omega; \mathbb{R}^n)$ is the continuous linear operator defined in Theorem 1*.*

Sketch of proof. (i) Assume first that a vector field $\boldsymbol{u} = (u_i) \in H^1(\Omega; \mathbb{R}^n)$ satisfies $\frac{1}{2}(\partial_j u_i + \partial_i u_j) = e_{ij}$ in $L^2(\Omega)$, and let there be given a vector field $\boldsymbol{\varphi} = (\varphi_i) \in \mathcal{D}_1(\Omega; \mathbb{R}^n)$. Define the functions $a_{ij} = -a_{ij} := \frac{1}{2}(\partial_j u_i - \partial_i u_j) \in L^2(\Omega)$, so that $\partial_j u_i = e_{ij} + a_{ij}$. Since each component φ_i of the vector field $\boldsymbol{\varphi}$ belongs to the space $L_0^2(\Omega)$, Theorem 1 shows that each vector field $\boldsymbol{T}\varphi_i = (T_j \varphi_i) \in H_0^1(\Omega; \mathbb{R}^n)$ satisfies $-\partial_j (T_j \varphi_i) = \varphi_i$ in $L^2(\Omega)$. Consequently,

$$\langle u_i, \varphi_i \rangle = \langle e_{ij}, T_i \varphi_j \rangle + \frac{1}{2} \langle a_{ij}, T_j \varphi_i - T_i \varphi_j \rangle.$$

We next prove that each function $(T_j \varphi_i - T_i \varphi_j) \in H_0^1(\Omega)$ also belongs to the space $L_0^2(\Omega)$. Consequently, $T_j \varphi_i - T_i \varphi_j = -\partial_k T_k (T_j \varphi_i - T_i \varphi_j)$. We also note that

$$\partial_k a_{ij} = \frac{1}{2}(\partial_{jk} u_i - \partial_{ik} u_j) = -\partial_i e_{kj} + \partial_j e_{ki} \quad \text{in } H^{-1}(\Omega),$$

so that we obtain

$$\langle a_{ij}, T_j \varphi_i - T_i \varphi_j \rangle = 2 \langle e_{ij}, \partial_k [T_i (T_j \varphi_k - T_k \varphi_j)] \rangle.$$

Therefore, relations (14) are established.

(ii) Assume next that a vector field $\boldsymbol{u} = (u_i) \in H^1(\Omega; \mathbb{R}^n)$ satisfies relations (14). Let then a matrix field $\boldsymbol{\psi} = (\psi_{ij}) \in \mathcal{D}(\Omega; \mathbb{S}^n)$ be given. We first prove that $(\partial_j \psi_{ij})_{i=1}^n \in \mathcal{D}_1(\Omega; \mathbb{R}^n)$ and that, by (14),

$$\frac{1}{2} \langle \partial_j u_i + \partial_i u_j, \psi_{ij} \rangle = -\langle e_{ij}, T_i (\partial_k \psi_{jk}) \rangle + \langle \partial_k e_{ij} - \partial_j e_{ik}, T_i (T_j (\partial_l \psi_{kl})) \rangle.$$

We next observe that the Saint Venant compatibility conditions with little regularity (6) may be rewritten as

$$\partial_l h_{jki} = \partial_i h_{jkl} \quad \text{in } H^{-2}(\Omega), \quad \text{where } h_{jki} = -h_{kji} := \partial_k e_{ji} - \partial_j e_{ki} \in H^{-1}(\Omega).$$

The Poincaré lemma with little regularity (Theorem 2) therefore shows that there exist functions $p_{jk} \in L^2(\Omega)$, each one being unique up to an additive constant, such that $\partial_i p_{jk} = h_{jki} = \partial_k e_{ij} - \partial_j e_{ik}$ in $H^{-1}(\Omega)$. Since $\partial_i (p_{jk} + p_{kj}) = h_{jki} + h_{kji} = 0$, these additive constants can be adjusted in such a way that $p_{jk} + p_{kj} = 0$ in $L^2(\Omega)$. Consequently,

$$\langle \partial_k e_{ij} - \partial_j e_{ik}, T_i (T_j (\partial_l \psi_{kl})) \rangle = -\frac{1}{2} \langle p_{jk}, \partial_i [T_i (T_j (\partial_l \psi_{kl}) - T_k (\partial_l \psi_{jl}))] \rangle.$$

We then show that each function $(T_j (\partial_l \psi_{kl}) - T_k (\partial_l \psi_{jl}))$ (i.e., for each $j = 1, \ldots, n$ and each $k = 1, \ldots, n$) also belongs to the space $L_0^2(\Omega)$. As a result,

$$\langle \partial_k e_{ij} - \partial_j e_{ik}, T_i (T_j (\partial_l \psi_{kl})) \rangle = \langle p_{jk}, T_j (\partial_l \psi_{kl}) \rangle,$$

so that

$$\frac{1}{2}\langle \partial_j u_i + \partial_i u_j, \psi_{ij}\rangle = \langle e_{ij}, \psi_{ij}\rangle + \big\langle p_{jk} - e_{jk}, \psi_{jk} + T_j(\partial_l \psi_{kl})\big\rangle,$$

since $\langle p_{jk}, \psi_{jk}\rangle = 0$. Noting that the functions $q_{jk} := p_{jk} - e_{jk} \in L^2(\Omega)$ satisfy $\partial_l q_{jk} = \partial_j q_{lk}$ in $H^{-1}(\Omega)$, we again resort to the Poincaré lemma with little regularity (Theorem 2) to conclude that there exist functions $v_k \in H^1(\Omega)$, each one being unique up to an additive constant, such that $q_{jk} = \partial_j v_k = p_{jk} - e_{jk}$ in $L^2(\Omega)$. Consequently,

$$\big\langle p_{jk} - e_{jk}, \psi_{jk} + T_j(\partial_l \psi_{kl})\big\rangle = -\big\langle v_k, \partial_j \psi_{jk} + \partial_j T_j(\partial_l \psi_{kl})\big\rangle,$$

since $(\psi_{jk} + T_j(\partial_l \psi_{kl})) \in H_0^1(\Omega)$. But the definition of the operators T_j and the symmetries $\psi_{kl} = \psi_{lk}$ together imply that

$$-\partial_j T_j(\partial_l \psi_{kl}) = \partial_l \psi_{kl} = \partial_j \psi_{jk}.$$

Combining the above relations, we are thus left with

$$\frac{1}{2}\langle \partial_j u_i + \partial_i u_j, \psi_{ij}\rangle = \langle e_{ij}, \psi_{ij}\rangle.$$

Since this relation holds for any matrix field $\boldsymbol{\psi} = (\psi_{ij}) \in \mathcal{D}(\Omega; \mathbb{S}^n)$, it follows that $\frac{1}{2}(\partial_j u_i + \partial_i u_j) = e_{ij}$ in $L^2(\Omega)$, as announced. □

We also show that the solution $\boldsymbol{u} = (u_i)$ to the equations $\frac{1}{2}(\partial_j u_i + \partial_i u_j) = e_{ij}$ in $L^2(\Omega)$ can be found by solving a *variational problem* (cf. (16) below), which satisfies all the assumptions of the *Lax–Milgram lemma* (as is easily seen). Note that both Theorems 5 and 6 have direct applications to *intrinsic elasticity*; cf. [7].

Theorem 6. *Let Ω be a simply-connected domain in $\mathbb{R}^n$, let the space $L_1^2(\Omega; \mathbb{R}^n)$ be defined as*

$$L_1^2\big(\Omega; \mathbb{R}^n\big) := \left\{ \boldsymbol{v} = (v_i) \in L^2(\Omega; \mathbb{R}^n);\ \int_\Omega v_i \,\mathrm{d}x = \int_\Omega (x_j v_i - x_i v_j)\,\mathrm{d}x = 0 \right\}, \tag{15}$$

and let there be given functions $e_{ij} = e_{ji} \in L^2(\Omega)$ that satisfy the Saint Venant compatibility conditions with little regularity (6)*.*

Then the variational problem: Find a vector field $(u_i) \in L_1^2(\Omega; \mathbb{R}^n)$ such that

$$\langle u_i, v_i\rangle = \big\langle e_{ij}, T_i v_j + \partial_k\big[T_i(T_j v_k - T_k v_j)\big]\big\rangle \quad \text{for all } (v_i) \in L_1^2\big(\Omega; \mathbb{R}^n\big), \tag{16}$$

has a unique solution. Besides, (u_i) is in fact in the space $H^1(\Omega; \mathbb{R}^n)$ and is a particular solution to the equations $\frac{1}{2}(\partial_j u_i + \partial_i u_j) = e_{ij}$ in $L^2(\Omega)$.

Finally, we show that, when the data are smooth enough, the *Cesàro–Volterra formula with little regularity reduces to the classical Cesàro–Volterra formula*.

Note that the proof of *relation* (17) *below, which only involves the functions e_{ij}, does not use that its left-hand side is also given by $\langle u_i, \varphi_i\rangle$*, by Theorem 5 (otherwise this information would immediately provide a "proof" of (17), through the expression of $u_i(x)$ given by the classical Cesàro–Volterra formula).

Theorem 7. *Let the assumptions be those of Theorem* 5*, the functions $e_{ij} = e_{ji} \in L^2(\Omega)$ being in addition assumed to be in the space $\mathcal{C}^1(\Omega) \cap H^1(\Omega)$, and let the operator $(T_i): L_0^2(\Omega) \to H_0^1(\Omega; \mathbb{R}^n)$ be that defined in Theorem* 1*.*

Fix a point $x_0 \in \Omega$, and, given any point $x \in \Omega$, let $\gamma(x)$ be any path of class $\mathcal{C}^1$ contained in Ω and joining x_0 to x. Then the right-hand side of the Cesàro–Volterra formula with little regularity (14) *can be rewritten in this case as*

$$\begin{aligned}&\big\langle e_{ij}, T_i\varphi_j + \partial_k\big[T_i(T_j\varphi_k - T_k\varphi_j)\big]\big\rangle \\ &\quad = \int_\Omega \left[\int_{\gamma(x)} \big\{ e_{ij}(y) + \big(\partial_k e_{ij}(y) - \partial_i e_{kj}(y)\big)(x_k - y_k) \big\}\,\mathrm{d}y_j \right] \varphi_i(x)\,\mathrm{d}x\end{aligned} \tag{17}$$

for all $(\varphi_i) \in \mathcal{D}_1(\Omega; \mathbb{R}^n)$.

Relations (17) *in turn imply that any vector field* $(u_i) \in H^1(\Omega; \mathbb{R}^n)$ *that satisfies the Cesàro–Volterra formula with little regularity* (14) *is also given by*

$$u_i(x) = \int_{\gamma(x)} \left\{ e_{ij}(y) + \left(\partial_k e_{ij}(y) - \partial_i e_{kj}(y)\right)(x_k - y_k) \right\} \mathrm{d}y_j, \quad x \in \Omega, \tag{18}$$

up to the addition of a vector field of the form $x \in \Omega \mapsto \boldsymbol{a} + \boldsymbol{A}\boldsymbol{o}\boldsymbol{x}$ *for some* $\boldsymbol{a} \in \mathbb{R}^n$ *and* $\boldsymbol{A} \in \mathbb{A}^n$. *Besides,* $(u_i) \in \mathcal{C}^2(\Omega; \mathbb{R}^n)$ *in this case.*

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