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Partial Differential Equations

Some inverse stability results for the bistable reaction–diffusion equation using Carleman inequalities

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Abstract

We consider the bistable equation $v_t - \Delta v = f(v,x)$, $f(v,x) = a(x)v(1-v)(v-\alpha(x))$ with homogeneous Neumann boundary conditions in a bounded domain $\Omega \subset \mathbb{R}^3$ with regular boundary. For this equation, we prove Lipschitz stability for the inverse problem of recovering parameters a and α from measurements of v in $(0,T) \times \omega$, where ω is an arbitrary nonempty open subset of Ω and measurements of $v(t_0)$ in the whole domain Ω at some positive time t_0 such that $0 < t_0 < T$. The result is based in some suitable global Carleman estimate for the nonlinear problem. ***To cite this article: M. Boulakia et al., C. R. Acad. Sci. Paris, Ser. I 347 (2009).***

Résumé

Quelques résultats de stabilité inverse à partir d'inégalités de Carleman pour l'équation bistable. Dans un domaine $\Omega \subset \mathbb{R}^3$ borné de frontière régulière, nous considérons l'équation bistable $v_t - \Delta v = f(v,x)$, $f(v,x) = a(x)v(1-v)(v-\alpha(x))$ complétée par des conditions de Neumann homogène au bord. Pour cette équation, nous prouvons un résultat de stabilité lipschitzienne pour le problème inverse qui consiste à identifier les paramètres a et α à partir de mesures de v sur $(0,T) \times \omega$, où $\omega \subset \Omega$ est un ouvert non vide quelconque et des mesures de $v(t_0)$ dans tout le domaine Ω avec t_0 tel que $0 < t_0 < T$. Le résultat est basé sur une inegalité de Carleman globale pour le problème non linéaire. ***Pour citer cet article : M. Boulakia et al., C. R. Acad. Sci. Paris, Ser. I 347 (2009).***

1. Introduction

Let $\Omega \subset \mathbb{R}^3$ be a bounded connected open set with regular boundary $\partial\Omega$ ($C^{2+\varepsilon}$, $\varepsilon > 0$). For $T > 0$, we define $Q = (0,T) \times \Omega$ and $\Sigma = (0,T) \times \partial\Omega$. We are interested in the identification of the nonlinear term in the bistable equation:

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$$\begin{cases} v_t - \Delta v = f(v,x) & \text{in } Q, \\ \nabla v \cdot n = 0 \quad \text{on } \Sigma, \quad v(0) = v_0 & \text{in } \Omega, \end{cases} \tag{1}$$

where $f(v,x) = a(x)v(1-v)(v-\alpha(x))$, from internal measurements. More precisely, we are interested in the identification of the parameters a and α that are supposed to satisfy

$$a \in W^{1,\infty}(\Omega), \quad \exists a_0, a_1 \in \mathbb{R} \text{ such that } 0 < a_0 \leqslant a(x), \quad \text{a.e. } x \in \Omega \text{ and } \|a\|_{W^{1,\infty}(\Omega)} \leqslant a_1, \tag{2}$$

$$\alpha \in L^\infty(\Omega), \quad \exists \alpha_0, \alpha_1 \in \mathbb{R} \text{ such that } 0 < \alpha_0 \leqslant \alpha(x) \leqslant \alpha_1 < 1, \quad \text{a.e. } x \in \Omega. \tag{3}$$

This model can be seen as a simple model for the propagation of a normalized voltage u in an insulated heart [7] so in the following we suppose that $u_0 \in (0,1)$. For a list of other related inverse problems of this kind see [3] and the references therein.

2. Main results

Theorem 2.1. *Let $T > 0$ and let us consider a nonempty open set ω included in Ω.*

(i) *We suppose that a satisfy (2) and α satisfy (3). Let v be the solution of (1) with initial condition $v_0 \in L^2(\Omega)$ and let $\bar{v}$ be another solution of the same system with initial condition $\bar{v}_0 \in H^2(\Omega)$, then for all $T' \in (0,T)$ there exists $C > 0$ depending on $\bar{v}$ such that*

$$\left\| v(T') - \bar{v}(T') \right\|_{L^2(\Omega)} \leqslant C\left(\|v - \bar{v}\|_{L^2(0,T;L^2(\omega))} + \|v - \bar{v}\|^2_{L^4(0,T;L^4(\omega))} \right).$$

(ii) *Let $v_0 \in H^2(\Omega)$ and $\bar{v}_0 \in H^4(\Omega)$, a and $\bar{a}$ satisfy (2) and α and $\bar{\alpha}$ satisfy (3). We denote by $(p, \bar{p}) = (a, \bar{a})$ or $(\alpha, \bar{\alpha})$, and by $\bar{v} = v(\bar{v}_0, \bar{p})$ and $v = v(v_0, p)$ the corresponding solutions of (1). Let us assume that*

$$\left| \frac{\partial f}{\partial p}\big(\bar{v}(x,t_0)\big) \right| \geqslant r_0 > 0 \quad \textit{for some } t_0 \in (0,T) \textit{ and for all } x \in \Omega. \tag{4}$$

Then there exists $C > 0$ such that $\|p - \bar{p}\|_{L^2(\Omega)} \leqslant C N_{T,\omega}(v - \bar{v})$, where $N_{T,\omega}(u) = \|u\|_{H^1(0,T;L^2(\omega))} + \|u\|^2_{L^4(0,T;L^4(\omega))} + \|u(t_0)\|_{H^2(\Omega)} + \|u(t_0)\|^3_{L^6(\Omega)}$.

Sketch of the proof of Theorem 2.1. Introducing $u = v - \bar{v}$ and using the exact expansion

$$f(v,x) - f(\bar{v},x) = -a(x)u^3 + g(\bar{v},u,x), \qquad g(\bar{v},u,x) = f'(\bar{v},x)u + \frac{f''(\bar{v},x)}{2}u^2,$$

the stability results (i) and (ii) can be reduced to the study of a Carleman inequality for the problem

$$\begin{cases} u_t - \Delta u + a u^3 = G & \text{in } Q, \\ \nabla u \cdot n = 0 \quad \text{on } \Sigma, \quad u(0) = u_0 & \text{in } \Omega, \end{cases} \tag{5}$$

where $G = g(\bar{v},u,x) + q(x)\overline{R}(t,x)$ a.e. $(t,x) \in Q$. In case (i) $q = 0$, and in case (ii) $q = p - \bar{p}$ and $\overline{R} = \frac{\partial f}{\partial p}(\bar{v})$ is the derivative of f with respect to the corresponding parameter evaluated at the reference trajectory $\bar{v}$, i.e.

$$\overline{R} = \bar{v}(1-\bar{v})\big(\bar{v} - \alpha(x)\big) \quad \text{if } p = a(x). \quad \big(\text{resp. } \overline{R} = -a(x)\bar{v}(1-\bar{v}) \quad \text{if } p = \alpha(x).\big) \tag{6}$$

The estimate in the case (i) can be derived directly by applying the estimates on u and u_t given by the Carleman inequality in Theorem 3.1.

In order to obtain estimate (ii), we follow the Bukhgeim–Klibanov approach [1]. We first consider $w = u_t$ and we write the equation satisfied by w as a heat equation $w_t - \Delta w = h$ in Q where $h = G_t - 3au^2w$. We supose to simplify that $0 < t_0 \leqslant T/2$ but the result still holds if $0 < t_0 < T$. Let us denote $T_0 = 2t_0 \leqslant T$ and $Q_0 = (0,T_0) \times \Omega$. We apply the usual Carleman estimate for the heat equation on $(0,T_0)$ and we obtain, for s large enough

$$\int_\Omega \rho(t_0)^2 \left|w(t_0)\right|^2 \mathrm{d}x \leqslant Cs^2 \iint_{(0,T_0)\times\omega} \rho^2|w|^2 \,\mathrm{d}x\,\mathrm{d}t + \frac{C}{s}\iint_{Q_0} \rho^2|h|^2 \,\mathrm{d}x\,\mathrm{d}t,$$

where the weight ρ is precisely defined in Section 3 by (8) with T replaced by T_0. Using (5), we have $w(t_0) = G(t_0) + \Delta u(t_0) - au^3(t_0)$. Thus, according to the definition of G, Proposition 2.2 and thanks to hypothesis (4) on $\overline{R}(t_0)$, we deduce that, for s large enough,

$$\int_{\Omega} \rho(t_0)^2 |q|^2 \,\mathrm{d}x \leqslant C s^2 N_{T_0,\omega}(u)^2 + \frac{C}{s} \iint_{Q_0} \rho^2 |h|^2 \,\mathrm{d}x\,\mathrm{d}t. \tag{7}$$

By definition of h, thanks to Proposition 2.2 which asserts that $v \in L^\infty(Q)$ and $\bar{v} \in W^{1,\infty}(0,T;L^\infty(\Omega))$, the last term is estimated by

$$\frac{C}{s} \iint_{Q_0} \rho^2 |h|^2 \,\mathrm{d}x\,\mathrm{d}t \leqslant \frac{C}{s} \iint_{Q_0} \rho^2 |w|^2 \,\mathrm{d}x\,\mathrm{d}t + \frac{C}{s} \iint_{Q_0} \rho^2 \big(|u|^2 + |u|^4\big) \,\mathrm{d}x\,\mathrm{d}t + \frac{C}{s} \iint_{Q_0} \rho^2 |q|^2 \,\mathrm{d}x\,\mathrm{d}t.$$

For the first term in the right-hand side, we apply a second time the Carleman inequality for the heat equation (we see that it is capital to have C/s in factor) and the second term is estimated thanks to the specific Carleman inequality of Theorem 3.1 on $(0,T_0)$. Finally, once the obtained inequality is injected in (7), the last term can be absorbed for s large enough by the left-hand side of (7) since $\rho \leqslant \rho(t_0)$ on $(0,T_0)$ thanks to the choice of $T_0 = 2t_0$. This ends the proof of (ii).

Our proof relies on the following standard regularity results of the solution of problem (5). Notice that the cubic nonlinear term does not affect the standard regularity for the heat equation.

Proposition 2.2. *We suppose that a and α satisfy* (2) *and* (3). *If v_0 belongs to $L^2(\Omega)$, system* (1) *admits a solution $v \in L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H^1(\Omega)) \cap L^4(0,T;L^4(\Omega))$.*

If $v_0 \in H^2(\Omega)$, system (1) *admits a solution v in $W^{1,\infty}(0,T;L^2(\Omega)) \cap H^1(0,T;H^1(\Omega)) \cap L^\infty(0,T;H^2(\Omega))$ and the norm of v in this space is bounded by a constant depending on the norm of v_0 in $H^2(\Omega)$, a_0 and a_1.*

Moreover, if $v_0 \in H^4(\Omega)$, v belongs to $W^{2,\infty}(0,T;L^2(\Omega)) \cap H^2(0,T;H^1(\Omega)) \cap W^{1,\infty}(0,T;H^2(\Omega))$ and the norm of v in this space is bounded by a constant depending on the norm of v_0 in $H^4(\Omega)$, a_0 and a_1.

The following proposition shows that there exist trajectories $\bar{v}$ satisfying hypothesis (4):

Proposition 2.3. *If $v_0 \in H^4(\Omega)$ is such that $v_0 \in (0,1)$ or $(0,\alpha_0)$ or $(\alpha_1,1)$ then $v(x,t)$ solution of* (1) *lies strictly on the same interval for $(x,t) \in \overline{\Omega} \times [0,T]$.*

We use the strong comparison principle (see [6] and [2]): let $Lu = u_t - \Delta u$ and $\underline{u}$, $\bar{u}$ super and subsolutions such that $L\underline{u} \leqslant f(\underline{u})$ and $L\bar{u} \geqslant f(\bar{u})$ in Q. If $\underline{u}(0) < \bar{u}(0)$ in Ω and $\frac{\partial \underline{u}}{\partial n} \leqslant \frac{\partial \bar{u}}{\partial n}$ on Σ, then either $\underline{u} \equiv \bar{u}$ or $\underline{u} < \bar{u}$ in Q. This comparison result is obtained by applying the strong maximum principle to $u = \bar{u} - \underline{u}$ satisfying $Lu - cu \geqslant 0$ with a bounded $c = (f(\bar{u}) - f(\underline{u}))/(\bar{u} - \underline{u})$. Now, for instance, if $v_0 \in (0,\alpha_0)$ the principle is applied twice with $\underline{u} = 0$, $\bar{u} = v$ and $\underline{u} = v$, $\bar{u} = \alpha_0$ to obtain that $0 < v(t,x) < \alpha_0$ for all $(t,x) \in Q$. To see that this is also true on Σ, we use the Hopf Lemma: if $v(t_0,x_0) = 0$ for some $t_0 \in (0,T)$ and $x_0 \in \partial\Omega$, then v attains its minimum on the boundary and in this case either $v = 0$ in $[0,t_0] \times \Omega$ or $\frac{\partial v}{\partial n}(t_0,x_0) > 0$. This leads to a contradiction with $v(0) = v_0 > 0$ in Ω or $\frac{\partial v}{\partial n} = 0$ on Σ respectively. In the same way, we obtain a contradiction if $v(t_0,x_0) = \alpha_0$ for some $t_0 \in (0,T)$ and $x_0 \in \partial\Omega$.

3. Global Carleman inequality for problem (5)

As for the linear heat equation [5], we define a function ψ in Ω such that $\psi \in C^2(\overline{\Omega})$, $\psi > 0$ in Ω, $\psi = 0$ on $\partial\Omega$, $|\nabla\psi| > 0$ in $\overline{\Omega \setminus \omega'}$, where $\omega' \Subset \omega$ is a nonempty open set. The existence of such function has been proved in [5]. We define, for all $\lambda > 0$ and $s > 0$ the following weights on Q

$$\varphi(\psi) = \frac{e^{2\lambda\|\psi\|_\infty} - e^{\lambda\psi(x)}}{t(T-t)}, \qquad \eta(\psi) = \frac{e^{\lambda\psi(x)}}{t(T-t)}, \qquad \rho(\psi) = e^{-s\varphi(\psi)}, \tag{8}$$

the weighted global energy $I(\psi)$ and the weighted local function of observations $N_{T,\omega,\psi}$ are defined by:

$$I(\psi)=\iint_Q \rho^2\left(\frac{1}{s\eta}\left(|u_t|^2+|\Delta u|^2\right)+|u|^6+s\lambda^2\eta|\nabla u|^2+s^3\lambda^4\eta^3|u|^2+s^2\lambda^2\eta^2|u|^4\right)\mathrm{d}x\,\mathrm{d}t,$$

$$N_{T,\omega,\psi}(u)=\iint_{(0,T)\times\omega}\rho^2\left(s^3\lambda^4\eta^3|u|^2+s^2\lambda^2\eta^2|u|^4\right)\mathrm{d}x\,\mathrm{d}t+\iint_Q \rho^2|G|^2\,\mathrm{d}x\,\mathrm{d}t.$$

Theorem 3.1. *There exists $\bar\lambda$ and C only depending on Ω and ω such that, for any $\lambda\geqslant\bar\lambda$, $s\geqslant\bar s=C(\Omega,\omega,T,a_0,a_1)\times e^{2\lambda\|\psi\|_\infty}$, for any $G\in L^2(Q)$ and $u_0\in L^2(\Omega)$, the solution u of* (5) *satisfies*

$$I(\psi)\leqslant C N_{T,\omega,\psi}(u). \tag{9}$$

Corollary 3.2. *Let u be a weak solution of* (5) *with $G=0$. If $u=0$ in $(0,T)\times\omega$, then $u=0$ in Q.*

Remark 1. If one replaces Neumann boundary conditions by homogeneous Dirichlet boundary conditions in (5), Theorem 3.1 still holds and is obtained more easily since most of boundary terms when obtaining the Carleman inequality vanish. But here we are interested in Neumann boundary conditions.

Remark 2. A second Carleman inequality can be obtained with boundary observations using a weight $\psi\in C^2(\overline{\Omega})$ such that $\psi=0$ on $\partial\Omega\setminus\gamma$, $\psi>0$ on γ', $|\nabla\psi|>0$ in $\overline{\Omega}$, where $\gamma'\Subset\gamma\subset\partial\Omega$. Note that in this case we have to control some boundary terms involving the tangential derivative $\frac{\partial u}{\partial\tau}$ along γ. This can be done by using an interpolation argument and the elliptic regularity. The unique continuation property of Corollary 3.2 is still true if we assume that $u=0$ in $(0,T)\times\gamma$.

Sketch of the proof of Theorem 3.1. We define $w=e^{-s\varphi}u$. Then, if u satisfies (5), w is solution of $P^1_\psi(w)+P^2_\psi(w)=H_\psi(w)$ where $P^1_\psi(w)=-\Delta w-s^2\lambda^2\eta^2|\nabla\psi|^2w+s\varphi_t w+\frac{3a}{4}e^{2s\varphi}w^3$, $P^2_\psi(w)=w_t+2s\lambda\eta\nabla\psi\cdot\nabla w+2s\lambda^2|\nabla\psi|^2\eta w+\frac{a}{4}e^{2s\varphi}w^3$, $H_\psi(w)=e^{-s\varphi}G(e^{s\varphi}w)-s\lambda\eta\Delta\psi w+s\lambda^2|\nabla\psi|^2\eta w$ and, as in [5], we estimate $(P^1_\psi,P^2_\psi)_{L^2(Q)}$ from below.

Note that the decomposition is similar to the standard decomposition for the heat equation [5] except for the cubic terms in w^3. The integrals coming from the nonlinear terms in P^1_ψ, P^2_ψ give the terms in $|u|^4$ and $|u|^6$ in $I(\psi)$ and the other additional terms can be absorbed by the dominating positive terms.

Since we consider Neumann boundary conditions, we also have to deal with the boundary integrals. The idea (we refer for instance to [5] and [4]) is to do the calculations for $w(\psi)$ and also for $w(-\psi)$ and then sum up the two inequalities. This allows one to cancel the boundary integrals and to obtain the inequality $I(\psi)+I(-\psi)\leqslant C(N_{T,\omega,\psi}(u)+N_{T,\omega,-\psi}(u))$. Thus, since $I(\psi)\leqslant I(\psi)+I(-\psi)$ and $N_{T,\omega,-\psi}(u)\leqslant N_{T,\omega,\psi}(u)$, we obtain (9).

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