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COMPTES RENDUS MATHEMATIQUE

Lie Algebras/Probability Theory

Radial Dunkl processes: Existence, uniqueness and hitting time

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Abstract

We give shorter proofs of the following known results: the radial Dunkl process associated with a reduced system and a strictly positive multiplicity function is the unique strong solution for all times t of a stochastic differential equation with a singular drift, the first hitting time of the Weyl chamber by a radial Dunkl process is finite almost surely for small values of the multiplicity function. The proof of the first result allows one to give a positive answer to a conjecture announced by Gallardo–Yor while that of the second shows that the process hits almost surely the wall corresponding to the simple root with a small multiplicity value. ***To cite this article: N. Demni, C. R. Acad. Sci. Paris, Ser. I 347 (2009).***

Résumé

Processus de Dunkl radial : Existence, unicité et temps d'atteinte. On donne de courtes preuves des résultats suivants : le processus de Dunkl radial associé à un système de racines réduit et une fonction de multiplicité strictement positive est l'unique solution forte d'une équation différentielle stochastique à dérive singulière pour tout temps t, le temps d'atteinte de la frontière de la chambre de Weyl est fini presque sûrement pour les petites valeurs de la fonction de multiplicité. La preuve du premier résultat permet de donner une réponse positive à une conjecture de Gallardo–Yor, alors que celle du deuxième résultat montre que le processus touche précisemment le mur correspondant à la racine simple pour laquelle la valeur de la multiplicité est suffisamment petite. ***Pour citer cet article : N. Demni, C. R. Acad. Sci. Paris, Ser. I 347 (2009).***

Let $(V, \langle,\rangle)$ be a finite real Euclidean space of dimension m and let R be a reduced root system with positive and simple systems R_+ and S [4]. Let W be the reflection groups and let k be a positive-valued multiplicity function. Let X denote the Dunkl process associated with R and k and let X^W, known as the radial Dunkl process, denote its projection on the closure of the Weyl chamber $\overline{C}$ [3]. We first give a shorter proof to the following statement (see [7] and [3, p. 170] for other proofs):

Theorem 1. *X^W is the unique strong solution of*

$$\mathrm{d}Y_t = \mathrm{d}B_t - \nabla\Phi(Y_t)\,\mathrm{d}t, \quad Y_0 \in \overline{C},\ t \geqslant 0,$$

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where

$$\Phi(x) := -\sum_{\alpha\in R_+} k(\alpha)\ln\big(\langle\alpha, x\rangle\big),$$

B is a Brownian motion in V, $k(\alpha) > 0$ for all $\alpha \in S$ and Y is a continuous $\overline{C}$-valued process.

Proof. Theorem 2.2 in [2] shows that the SDE

$$\mathrm{d}Y_t = \mathrm{d}B_t - \nabla\Phi(Y_t)\,\mathrm{d}t + n(Y_t)\,\mathrm{d}L_t, \quad Y_0 \in \overline{C}, \tag{1}$$

where $n(x)$ belongs to the set of unitary inward normal vectors to C at $x \in V$ defined by

$$\big\langle x - a, n(x)\big\rangle \leqslant 0, \quad a \in \overline{C}, \tag{2}$$

and L is the boundary process satisfying:

$$\mathrm{d}L_t = \mathbf{1}_{\{Y_t\in\partial C\}}\,\mathrm{d}L_t,$$

has a unique strong solution for all $t \geqslant 0$. Moreover:

$$\mathbb{E}_y\left[\int_0^T \mathbf{1}_{\{Y_t\in\partial C\}}\,\mathrm{d}t\right] = 0,$$

$$\mathbb{E}_y\left[\int_0^T \big|\nabla\Phi(Y_t)\big|\,\mathrm{d}t\right] < \infty \tag{3}$$

for all $T > 0$ and $y \in \overline{C}$. All what we need is to prove that $(L_t)_{t\geqslant 0}$ vanishes. To proceed, we need two lemmas:

Lemma 1. *Set $\mathrm{d}G_t := n(Y_t)\,\mathrm{d}L_t$. Then, for all $\alpha \in R_+$,*

$$\mathbf{1}_{\{\langle Y_t,\alpha\rangle=0\}}\langle \mathrm{d}G_t, \alpha\rangle = 0.$$

Proof. It follows the same lines of the one written in [1] for particles system in the real line or equivalently to $R = A_{m-1}$ [4, p. 41]. □

Lemma 2. *Let $x \in \partial C$. Then $\langle n(x), \alpha\rangle \neq 0$ for some $\alpha \in S$ such that $\langle x, \alpha\rangle = 0$.*

Proof. Assume that $\langle n(x), \alpha\rangle = 0$ for all $\alpha \in S$ such that $\langle x, \alpha\rangle = 0$. The idea is to find a strictly positive constant ϵ such that $x - \epsilon n(x) \in \overline{C}$ then use the definition of inward normal vectors (see (2) above) to conclude that $n(x) = 0$. Our assumption implies that $\langle x, \alpha\rangle > 0$ for all $\alpha \in S$ such that $\langle n(x), \alpha\rangle \neq 0$. If such simple roots do not exist, that is, $\langle n(x), \alpha\rangle = 0$ for all $\alpha \in S$, then $x - \epsilon n(x) \in \overline{C}$ for all $\epsilon > 0$. Otherwise, if $\langle n(x), \alpha\rangle < 0$ for these simple roots, then $x - \epsilon n(x) \in \overline{C}$ for all $\epsilon > 0$. Finally, if none of these conditions is satisfied, choose

$$0 < \epsilon < \min_{\langle x,\alpha\rangle>0,\ \langle n(x),\alpha\rangle>0} \frac{\langle x, \alpha\rangle}{\langle n(x), \alpha\rangle},$$

to see that $x - \epsilon n(x) \in \overline{C}$. Substituting $a = x - \epsilon n(x)$ in (2) for the three alternatives, it then follows that $n(x)$ is the null vector, contradiction. □

Now we proceed to end the proof of Theorem 1. Lemma 2 asserts that

$$\{Y_t \in \partial C\} \subset \bigcup_{\alpha\in S}\big\{\langle Y_t, \alpha\rangle = 0,\ \big\langle n(Y_t), \alpha\big\rangle \neq 0\big\}$$

for all t. It follows that

$$0 \leqslant L_t \leqslant \sum_{\alpha \in S} \int_0^t \mathbf{1}_{\{\langle Y_s, \alpha\rangle = 0,\, \langle n(Y_s), \alpha\rangle \neq 0\}} \, \mathrm{d}L_s$$
$$= \sum_{\alpha \in S} \int_0^t \frac{1}{\langle n(Y_s), \alpha\rangle} \mathbf{1}_{\{\langle Y_s, \alpha\rangle = 0,\, \langle n(Y_s), \alpha\rangle \neq 0\}} \big\langle n(Y_s), \alpha\big\rangle \, \mathrm{d}L_s = 0$$

by Lemma 1. □

As a corollary of Theorem 1, we prove the validity of the following conjecture, announced in [3, p. 127]:

Corollary 1. *For all $x \in V$ and a strictly positive k, one has for all time t*

$$\sum_{s \leqslant t} |\Delta X_s| := \sum_{s \leqslant t} |X_s - X_{s-}| < \infty \quad \mathbb{P}_x \text{ almost surely},$$

where $|\cdot| := \langle \cdot, \cdot \rangle$.

Proof. Using the Lévy kernel [3, p. 123], it suffices to prove that

$$\mathbb{E}_x \left[\int_0^t \mathrm{d}s \sum_{\alpha \in R_+} \frac{k(\alpha)}{|\langle \alpha, X_s \rangle|} \right] < \infty.$$

However, using the semi group densities of X and X^W [3, pp. 120–122], one has

$$\int_0^t \mathrm{d}s \, \mathbb{E}_x \left[\sum_{\alpha \in R_+} \frac{k(\alpha)}{|\langle \alpha, X_s \rangle|} \right] = \int_0^t \mathrm{d}s \, \mathbb{E}_{x'} \left[\sum_{\alpha \in R_+} \frac{k(\alpha)}{\langle \alpha, X_s^W \rangle} \right]$$

where $x' \in \overline{C}$ is the unique representative of x lying in $\overline{C}$. The conjecture easily follows from (3). □

Our last result concerns the finiteness of the first hitting of the boundary ∂C of C:

$$T_0 := \inf\big\{t > 0,\ X_t^W \in \partial C\big\} = \inf_{\alpha \in S} \inf\big\{t > 0,\ \big\langle \alpha, X_t^W \big\rangle = 0\big\} := \inf_{\alpha \in S} T_\alpha.$$

Proposition 1. *Let $\alpha \in S$ such that $0 \leqslant k(\alpha) < 1/2$, then $T_0 < T_\alpha < \infty$ almost surely.*

Proof. Assume $k(\alpha) > 0$ for all $\alpha \in R$ and let $\alpha_0 \in S$. Using Theorem 1 and Itô's formula, one derives

$$\mathrm{d}\big\langle \alpha_0, X_t^W \big\rangle = |\alpha_0| \, \mathrm{d}\gamma_t + k_0 \frac{\|\alpha_0\|^2}{\langle \alpha_0, X_t^W \rangle} \, \mathrm{d}t + \sum_{\alpha \in R_+ \setminus \alpha_0} k(\alpha) \frac{\langle \alpha, \alpha_0 \rangle}{\langle \alpha, X_t^W \rangle} \, \mathrm{d}t.$$

Set $R = \bigcup_{j=1}^p R^j$, R^j, $1 \leqslant j \leqslant p$, being the orbits of R under the W-action and $R_+ = \bigcup_{j=1}^p R_+^j$. Then one has with $k_0 := k(\alpha_0)$:

$$\mathrm{d}\big\langle \alpha_0, X_t^W \big\rangle = |\alpha_0| \, \mathrm{d}\gamma_t + k_0 \frac{|\alpha_0|^2}{\langle \alpha_0, X_t^W \rangle} \, \mathrm{d}t + \sum_{j=0}^p \sum_{\alpha \in R_+^j \setminus \alpha_0} k(\alpha) \frac{\langle \alpha, \alpha_0 \rangle}{\langle \alpha, X_t^W \rangle} \, \mathrm{d}t.$$

If $\langle \alpha, \alpha_0 \rangle := a(\alpha) > 0$ then $\langle \sigma_0(\alpha), \alpha_0 \rangle = -a(\alpha)$ where $\sigma_0 := \sigma_{\alpha_0}$. Note that $\sigma_0(\alpha)$ and α lie in the same orbit and that $\sigma_0(\alpha) \in R_+ \setminus \alpha_0$ for $\alpha \in R_+ \setminus \alpha_0$ (Proposition 1.4 in [4]). Hence,

$$\mathrm{d}\big\langle \alpha_0, X_t^W \big\rangle = |\alpha_0| \, \mathrm{d}\gamma_t + k_0 \frac{|\alpha_0|^2}{\langle \alpha_0, X_t^W \rangle} \, \mathrm{d}t - \sum_{j=0}^p \sum_{\substack{\alpha \in R_+^j \setminus \alpha_0 \\ a(\alpha) > 0}} k(\alpha) \frac{a(\alpha) \langle \alpha - \sigma_0(\alpha), X_t^W \rangle}{\langle \alpha, X_t^W \rangle \langle \sigma_0(\alpha), X_t^W \rangle} \, \mathrm{d}t.$$

Furthermore,

$$\alpha - \sigma_0(\alpha) = 2\frac{\langle \alpha, \alpha_0\rangle}{\langle \alpha_0, \alpha_0\rangle}\alpha_0 \quad \text{so that} \quad \big\langle \alpha - \sigma_0(\alpha), X_t^W\big\rangle = 2a(\alpha)\frac{\langle \alpha_0, X_t^W\rangle}{|\alpha_0|^2}.$$

Consequently:

$$\mathrm{d}\big\langle \alpha_0, X_t^W\big\rangle = |\alpha_0|\,\mathrm{d}\gamma_t + k_0\frac{|\alpha_0|^2}{\langle \alpha_0, X_t^W\rangle}\,\mathrm{d}t + F_t\,\mathrm{d}t$$

where $F_t < 0$ on $\{T_{\alpha_0} = \infty\}$. Using Proposition 2.18, p. 293 and Exercise 2.19, p. 294 in [5], it follows that $\langle \alpha_0, X_t^W\rangle \leqslant Y^x_{|\alpha_0|^2 t}$ for all $t \geqslant 0$ on $\{T_{\alpha_0} = \infty\}$, where Y^x is a Bessel process of dimension $2k_0 + 1$ and starting at $Y_0 = x \geqslant \langle \alpha_0, X_0^W\rangle > 0$. However, this is impossible if $k_0 < 1/2$ since for, Y^x hits 0 almost surely [6, Ch. XI]. □

References

[1] E. Cépa, D. Lépingle, Diffusing particles with electrostatic repulsions, Probab. Theory Related Fields 107 (1997) 429–449.

[2] E. Cépa, D. Lépingle, Brownian particles with electrostatic repulsion on the circle: Dyson's model for unitary random matrices revisited, ESAIM: Probability and Statistics 5 (2001) 203–224.

[3] O. Chybiryakov, N. Demni, L. Gallardo, M. Rösler, M. Voit, M. Yor, Harmonic and Stochastic Analysis of Dunkl Processes, Travaux en Cours, Hermann, 2008.

[4] J.E. Humphreys, Reflections Groups and Coxeter Groups, vol. 29, Cambridge University Press, 2000.

[5] I. Karatzas, S.E. Shreve, Brownian Motion and Stochastic Calculus, 2nd ed., Springer-Verlag, New York, 1991.

[6] D. Revuz, M. Yor, Continuous Martingales and Brownian Motion, 3rd ed., Springer, 1999.

[7] B. Schapira, The Heckman–Opdam Markov processes, Probab. Theory Related Fields 138 (3–4) (2007) 495–519.