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Exact reachability for second-order integro-differential equations

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Abstract

In this Note we analyze a reachability problem for an integro-differential equation by using a harmonic analysis approach. ***To cite this article: P. Loreti, D. Sforza, C. R. Acad. Sci. Paris, Ser. I 347 (2009).***

Résumé

Atteignabilité exacte pour des équations intégro-différentielles du deuxième ordre. Dans cette Note on étudie un problème d'atteignabilité pour une équation intégro-différentielle du second ordre par une approche utilisant des techniques d'analyse harmonique. ***Pour citer cet article : P. Loreti, D. Sforza, C. R. Acad. Sci. Paris, Ser. I 347 (2009).***

Version française abrégée

Dans cette Note on étudie une équation intégro-différentielle ($T > 0$, donné, $0 < \beta < \eta$) :

$$u_{tt}(t,x) - u_{xx}(t,x) + \beta \int_0^t e^{-\eta(t-s)} u_{xx}(s,x)\,\mathrm{d}s = 0, \quad t \in (0,T),\ x \in (0,\pi), \tag{1}$$

pour des conditions initiales nulles,

$$u(0,x) = u_t(0,x) = 0, \quad x \in (0,\pi), \tag{2}$$

et des conditions aux limites,

$$u(t,x) = \begin{cases} 0 & x = 0, \\ g(t) & x = \pi. \end{cases} \tag{3}$$

Si on considère g comme une fonction de contrôle, le problème d'atteignabilité revient à démontrer qu'il existe $g \in L^2(0,T)$ telle que la solution faible de l'equation (1), sous les conditions limites (3), passe en un temps fini, de l'état nul à un état donné. Plus précisément, on adopte la même définition d'atteignabilité pour des systèmes à mémorie

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donnée dans la littérature par plusieurs auteurs, voir par exemple [8,5,6,10]. En effet, considérons le problème : étant donné, $T > 0$, $u_0 \in L^2(0,\pi)$, $u_1 \in H^{-1}(0,\pi)$, trouver $g \in L^2(0,T)$ telle que la solution faible u du problème (1)–(3) vérifie les conditions finales :

$$u(T,x) = u_0(x), \qquad u_t(T,x) = u_1(x), \quad x \in (0,\pi). \tag{4}$$

Notre but est d'établir ce résultat sans aucune hypothèse sur la petitesse du noyau de convolution, comme le suggère J.-L. Lions dans [8, p. 258].

On résout le problème posé par les méthodes de l'analyse harmonique [12] : on démontre des théorèmes du type Ingham [3], théorèmes que nous utilisons en combinaison avec la méthode HUM (Hilbert Uniqueness Method), introduite par J.-L. Lions dans [7].

1. Introduction

In this Note we analyze the integro-differential equation ($T > 0$ given, $0 < \beta < \eta$),

$$u_{tt}(t,x) - u_{xx}(t,x) + \beta \int_0^t e^{-\eta(t-s)} u_{xx}(s,x)\,\mathrm{d}s = 0, \quad t \in (0,T),\ x \in (0,\pi), \tag{5}$$

with null initial conditions,

$$u(0,x) = u_t(0,x) = 0, \quad x \in (0,\pi), \tag{6}$$

and boundary conditions,

$$u(t,x) = \begin{cases} 0 & x = 0, \\ g(t) & x = \pi. \end{cases} \tag{7}$$

If we regard g as a control function, our reachability problem consists in proving the existence of $g \in L^2(0,T)$ such that a weak solution of Eq. (5), subject to boundary conditions (7), moves from the null state to a given one in finite control time. To be more precise, we adopt the same definition of reachability problems for systems with memory given by several authors in the literature, see for example [8,5,6,10]. Indeed, we mean the following: given $T > 0$, $u_0 \in L^2(0,\pi)$ and $u_1 \in H^{-1}(0,\pi)$, find $g \in L^2(0,T)$ such that the weak solution u of problem (5)–(7) verifies the final conditions:

$$u(T,x) = u_0(x), \qquad u_t(T,x) = u_1(x), \quad x \in (0,\pi). \tag{8}$$

Our goal is to achieve such result without any smallness assumption on the convolution kernel, as suggested by J.-L. Lions in [8, p. 258].

We solve the problem by the harmonic analysis approach, see [12]. We prove Ingham type theorems [3] and use them to apply the HUM (Hilbert Uniqueness Method), introduced by J.-L. Lions in [7].

2. Hilbert Uniqueness Method

Given $(z_0, z_1) \in C_c^\infty(0,\pi) \times C_c^\infty(0,\pi)$, we consider the adjoint problem of (5):

$$z_{tt}(t,x) - z_{xx}(t,x) + \beta \int_t^T e^{-\eta(s-t)} z_{xx}(s,x)\,\mathrm{d}s = 0, \quad t \in (0,T),\ x \in (0,\pi), \tag{9}$$

with homogeneous boundary conditions,

$$z(t,0) = z(t,\pi) = 0, \quad t \in (0,T), \tag{10}$$

and final data,

$$z(T,x) = z_0(x), \qquad z_t(T,x) = z_1(x), \quad x \in (0,\pi). \tag{11}$$

Let ϕ be the unique solution of the problem,

$$\begin{cases} \phi_{tt}(t,x) - \phi_{xx}(t,x) + \beta \int_0^t e^{-\eta(t-s)} \phi_{xx}(s,x)\,\mathrm{d}s = 0, & t \in (0,T),\ x \in (0,\pi), \\ \phi(t,0) = 0, \quad \phi(t,\pi) = z_x(t,\pi) - \beta \int_t^T e^{-\eta(s-t)} z_x(s,\pi)\,\mathrm{d}s, & t \in (0,T), \\ \phi(0,x) = \phi_t(0,x) = 0, & x \in (0,\pi). \end{cases}$$

We define the linear operator:

$$\Psi(z_0, z_1) = \big(-\phi_t(T,\cdot), \phi(T,\cdot)\big), \quad (z_0,z_1) \in C_c^\infty(0,\pi) \times C_c^\infty(0,\pi).$$

The operator Ψ is well-defined, since ϕ is sufficiently regular. We have:

$$\big\langle \Psi(z_0,z_1), (z_0,z_1) \big\rangle = \int_0^T \left| z_x(t,\pi) - \beta \int_t^T e^{-\eta(s-t)} z_x(s,\pi)\,\mathrm{d}s \right|^2 \mathrm{d}t.$$

Therefore, we can introduce the semi-norm:

$$(z_0,z_1) \in C_c^\infty(0,\pi) \times C_c^\infty(0,\pi),$$
$$\big\|(z_0,z_1)\big\|_F := \left(\int_0^T \left| z_x(t,\pi) - \beta \int_t^T e^{-\eta(s-t)} z_x(s,\pi)\,\mathrm{d}s \right|^2 \mathrm{d}t \right)^{1/2}.$$

If $\|\cdot\|_F$ is a norm, then we can define a Hilbert space F as the completion of $C_c^\infty(0,\pi) \times C_c^\infty(0,\pi)$ for the norm $\|\cdot\|_F$. Moreover, the operator Ψ extends uniquely to a continuous operator, denoted again by Ψ, from F to the dual space F' in such a way that $\Psi : F \to F'$ is an isomorphism. So, if we prove that $F = H_0^1(0,\pi) \times L^2(0,\pi)$, then we can solve the reachability problem (5)–(8).

3. Main results

To prove that $\|\cdot\|_F$ is a norm and that $F = H_0^1(0,\pi) \times L^2(0,\pi)$, one can use the following two theorems.

Throughout, we consider functions of the type,

$$f(t) := \sum_{n=-\infty}^{\infty} \big(C_n e^{i\omega_n t} + R_n e^{r_n t} \big), \quad t \geqslant 0 \tag{12}$$

with $\omega_n, C_n \in \mathbb{C}$ and $r_n, R_n \in \mathbb{R}$ such that the sequences $\{\Im\omega_n\}$, $\{r_n\}$ are bounded and

$$\sum_{n=-\infty}^{\infty} |C_n|^2 < +\infty, \qquad \sum_{n=-\infty}^{\infty} |R_n|^2 < +\infty.$$

Let $T > 0$. We have an Ingham's type *inverse* inequality.

Theorem 1. *Let $\{\omega_n\}_{n\in\mathbb{Z}}$ and $\{r_n\}_{n\in\mathbb{Z}}$ be two sequences of pairwise distinct numbers such that $r_n \neq i\omega_m$ for any $n,m \in \mathbb{Z}$. Assume*

$$\Re\omega_n - \Re\omega_{n-1} \geqslant \gamma > 0, \quad \forall |n| \geqslant n',$$
$$\lim_{|n|\to\infty} \Im\omega_n = \alpha, \qquad r_n \leqslant -\Im\omega_n, \quad \forall |n| \geqslant n',$$
$$|R_n| \leqslant \frac{\mu}{|n|^\nu} |C_n|, \quad \forall |n| \geqslant n'; \qquad |R_n| \leqslant \mu |C_n|, \quad \forall |n| \leqslant n',$$

for some $n' \in \mathbb{N}$, $\alpha, M \in \mathbb{R}$, $\mu > 0$ and $\nu > 1/2$. Then, for any $T > 2\pi/\gamma$ we have:

$$c_1(T) \sum_{n=-\infty}^{\infty} |C_n|^2 \leqslant \int_0^T \big| f(t) \big|^2 \,\mathrm{d}t,$$

where $c_1(T)$ is a positive constant.

We have an Ingham's type *direct* inequality.

Theorem 2. *Assume*

$$\Re\omega_n - \Re\omega_{n-1} \geqslant \gamma > 0, \quad \forall |n| \geqslant n',$$
$$\lim_{|n|\to\infty} \Im\omega_n = \alpha,$$
$$|R_n| \leqslant \frac{\mu}{|n|^\nu}|C_n|, \quad \forall |n| \geqslant n'; \qquad |R_n| \leqslant \mu |C_n|, \quad \forall |n| \leqslant n',$$

for some $n' \in \mathbb{N}$, $\mu > 0$, $\nu > 1/2$ *and* $\alpha \in \mathbb{R}$. *Then, for any* $T > \pi/\gamma$ *we have*

$$\int_{-T}^{T} \left| f(t) \right|^2 \mathrm{d}t \leqslant c_2(T) \sum_{n=-\infty}^{\infty} |C_n|^2,$$

where $c_2(T)$ *is a positive constant.*

Main tools to prove the previous theorems are the following two results, which can be considered as Haraux's type theorems, see [2].

Proposition 3. *Let* $\{\omega_n\}_{n\in\mathbb{Z}}$ *be a sequence of pairwise distinct complex numbers such that*

$$\lim_{|n|\to\infty} |\omega_n| = +\infty$$

and let $\{r_n\}_{n\in\mathbb{Z}}$ *be a sequence of pairwise distinct real numbers such that* $r_n \neq i\omega_m$ *for any* $n, m \in \mathbb{Z}$. *Assume that there exists a finite set* $\mathcal{F}$ *of integers such that for any sequences* $\{C_n\}$ *and* $\{R_n\}$ *verifying* $C_n = R_n = 0$ *for any* $n \in \mathcal{F}$, *the estimates*

$$c_1' \sum_{n\notin\mathcal{F}} |C_n|^2 \leqslant \int_0^T \left| \sum_{n\notin\mathcal{F}} \left(C_n e^{i\omega_n t} + R_n e^{r_n t} \right) \right|^2 \mathrm{d}t \leqslant c_2' \sum_{n\notin\mathcal{F}} |C_n|^2$$

are satisfied for some constants $c_1', c_2' > 0$. *Then, there exists* $c_1 > 0$ *such that for any sequences* $\{C_n\}$ *and* $\{R_n\}$ *the estimate*

$$c_1 \sum_{n=-\infty}^{\infty} |C_n|^2 \leqslant \int_0^T \left| \sum_{n=-\infty}^{\infty} \left(C_n e^{i\omega_n t} + R_n e^{r_n t} \right) \right|^2 \mathrm{d}t$$

holds.

Proposition 4. *Assume that there exists a finite set* $\mathcal{F}$ *of integers such that for any sequences* $\{C_n\}$ *and* $\{R_n\}$ *verifying* $C_n = R_n = 0$ *for any* $n \in \mathcal{F}$, *the estimate*

$$\int_{-T}^{T} \left| \sum_{n\notin\mathcal{F}} \left(C_n e^{i\omega_n t} + R_n e^{r_n t} \right) \right|^2 \mathrm{d}t \leqslant c_2' \sum_{n\notin\mathcal{F}} |C_n|^2$$

is satisfied for some $c_2' > 0$. *Then, for any sequences* $\{C_n\}$ *and* $\{R_n\}$ *verifying*

$$|R_n| \leqslant \mu |C_n| \quad \text{for any } n \in \mathcal{F}$$

for some $\mu > 0$, *the estimate*

$$\int_{-T}^{T} \left| \sum_{n=-\infty}^{\infty} \left(C_n e^{i\omega_n t} + R_n e^{r_n t} \right) \right|^2 \mathrm{d}t \leqslant c_2 \sum_{n=-\infty}^{\infty} |C_n|^2,$$

holds for some $c_2 > 0$.

The proofs of these results are rather technical, so we refer to [9] for a detailed analysis. In particular, to prove the inverse inequality we introduce a family of operators, which will be needed to annihilate a finite number of terms in the Fourier series. Our operators are slightly different from those proposed in [2] and [4]. Given $\delta > 0$, $\omega \in \mathbb{C}$ and $r \in \mathbb{R}$ arbitrarily, we define the linear operators $I_{\delta,\omega}$ and $I_{\delta,\omega,r}$ as follows: for every continuous function $u:\mathbb{R} \to \mathbb{C}$ the function $I_{\delta,\omega}u:\mathbb{R} \to \mathbb{C}$ is given by the formula

$$I_{\delta,\omega}u(t) := u(t) - \frac{1}{\delta}\int_0^\delta e^{-i\omega s}u(t+s)\,\mathrm{d}s, \quad t \in \mathbb{R},$$

and

$$I_{\delta,\omega,r} := I_{\delta,\omega} \circ I_{\delta,-ir}.$$

For Ingham's type estimates, our results can be compared with those proved in [10], where functions of the type,

$$f(t) = \sum_{n=-\infty}^{\infty} \left(C_n e^{ir_n t} + C_n' e^{ir_n' t}\right), \quad t \geqslant 0$$

($r_n, r_n' \in \mathbb{R}$, $C_n, C_n' \in \mathbb{C}$) are considered. Our analysis is different from that of [10], because the admissible integral kernels are exponential functions. This class of kernels arises in linear viscoelasticity theory, such as in the analysis of Maxwell fluids or Poynting–Thomson solids, see e.g. [11].

Concerning Haraux's type estimates, in [4] functions of the type

$$f(t) = \sum_{n=-\infty}^{\infty} C_n e^{i\omega_n t}, \quad t \geqslant 0,$$

($\omega_n, C_n \in \mathbb{C}$) have been studied.

However, our analysis of the estimates changes completely with respect to that of cited papers, because the functions under study are different. Indeed, as we shall see in the next section, exponential kernels lead to a new form of the functions, see (12), where the exponents $i\omega_n$ also have a non-vanishing real part and some other real terms $R_n e^{r_n t}$ appear in the sum.

In addition, the choice of weight function is fundamental in this study and we borrow from [1] the idea of a different weight function with respect to the classical case [3].

Other papers related to our problem are [6] and [13,14], where the approach is not an Ingham type one.

4. An application

We end our analysis showing the spectral properties of our model, which allow us to put the solutions of problem (9)–(11) in the form (12).

We give our spectral analysis in an abstract setting. In the following X is a real Hilbert space with scalar product $\langle \cdot,\cdot \rangle$. Let $A : D(A) \subset X \to X$ be a self-adjoint positive linear operator on X with dense domain $D(A)$ and let $\{\lambda_j\}_{j\geqslant 1}$ be a strictly increasing sequence of eigenvalues for the operator A with $\lambda_j > 0$ and $\lambda_j \to \infty$ such that the sequence of the corresponding eigenvectors $\{e_j\}_{j\geqslant 1}$ constitutes a Hilbert basis for X.

For any $v_0 \in D(\sqrt{A})$ and $v_1 \in X$ there exists a unique weak solution $v \in C([0,\infty); D(\sqrt{A})) \cap C^1([0,\infty); X)$ of equation

$$v''(t) + Av(t) - \beta \int_0^t e^{-\eta(t-s)} Av(s)\,\mathrm{d}s = 0, \quad t \geqslant 0, \tag{13}$$

verifying the initial conditions

$$v(0) = v_0, \qquad v'(0) = v_1. \tag{14}$$

We have

$$v_0 = \sum_{j=1}^{\infty} \alpha_j e_j, \quad \alpha_j = \langle v_0, e_j \rangle, \qquad \sum_{j=1}^{\infty} \alpha_j^2 \lambda_j < \infty,$$
$$v_1 = \sum_{j=1}^{\infty} \gamma_j e_j, \quad \gamma_j = \langle v_1, e_j \rangle, \qquad \sum_{j=1}^{\infty} \gamma_j^2 < \infty.$$

To determine explicitly the solution, we write v in the form

$$v(t) = \sum_{j=1}^{\infty} f_j(t) e_j, \quad f_j(t) = \langle v(t), e_j \rangle.$$

We find that $f_j(t)$ is given by $f_j(t) = C_{1,j} e^{t\Lambda_{1,j}} + C_{2,j} e^{t\Lambda_{2,j}} + C_{3,j} e^{t\Lambda_{3,j}}$, where $\Lambda_{1,j}, C_{1,j} \in \mathbb{R}$ while $\Lambda_{k,j}, C_{k,j}$, $k = 2, 3$, are complex numbers. In addition,

$$\lim_{j\to\infty} \Lambda_{1,j} = \beta - \eta, \qquad \lim_{j\to\infty} |\Lambda_{2,j}| = +\infty, \qquad \lim_{j\to\infty} |\Lambda_{3,j}| = +\infty, \tag{15}$$

and there exists a positive constant c such that for any $j \in \mathbb{N}$ we have

$$\frac{|C_{1,j}|}{|C_{2,j}|} \leqslant \frac{c}{\lambda_j}, \qquad \frac{|C_{1,j}|}{|C_{3,j}|} \leqslant \frac{c}{\lambda_j}. \tag{16}$$

In conclusion, we prove that the solution $v(t)$ of Cauchy problem (13)–(14) can be written as

$$v(t) = \sum_{j=1}^{\infty} \left(C_{1,j} e^{t\Lambda_{1,j}} + C_{2,j} e^{t\Lambda_{2,j}} + C_{3,j} e^{t\Lambda_{3,j}} \right) e_j, \quad t \geqslant 0, \tag{17}$$

where $\Lambda_{k,j}$ and $C_{k,j}$ verify conditions (15) and (16) respectively.

Since we must estimate the L^2-norm in time of function v, in formula (17) we may skip the dependence on eigenvectors e_j, so the function to evaluate is of the type

$$f(t) := \sum_{j=1}^{\infty} \left(C_{1,j} e^{t\Lambda_{1,j}} + C_{2,j} e^{t\Lambda_{2,j}} + C_{3,j} e^{t\Lambda_{3,j}} \right), \quad t \geqslant 0.$$

It is then possible to show that f can be written in the form (12) where $C_n, \omega_n \in \mathbb{C}$ and $R_n, r_n \in \mathbb{R}$ verify the assumptions of our abstract results.

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