Partial Differential Equations

Restriction of toral eigenfunctions to hypersurfaces

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Abstract

Let $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$ be the d-dimensional flat torus. We establish for $d = 2, 3$ uniform upper and lower bounds on the restrictions of the eigenfunctions of the Laplacian to smooth hyper-surfaces with non-vanishing curvature. ***To cite this article: J. Bourgain, Z. Rudnick, C. R. Acad. Sci. Paris, Ser. I 347 (2009).***
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Résumé

Sur la restriction de fonctions propre du tore à des hyper-surfaces. Soit $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$ le tore plat d-dimensionnel. Pour $d = 2$ et $d = 3$, on établit des bornes supérieures et inférieures uniformes sur les restrictions des fonctions propres de l'opérateur de Laplace–Beltrami à des surfaces lisses de courbure non nulle. ***Pour citer cet article : J. Bourgain, Z. Rudnick, C. R. Acad. Sci. Paris, Ser. I 347 (2009).***
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Version française abrégée

Dans cette Note, nous étudions les restrictions de fonctions propres du tore plat $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$ à des hyper surfaces Σ compactes, lisses et de courbure non nulle. Dans ce contexte, nous améliorons certains résultats obtenus dans [1] (qui traite le cas général). Nous démontrons en particulier le suivant :

Théorème 1. *Soit $d = 2$ ou $d = 3$. Il existe des constantes $0 < c(\Sigma) < C(\Sigma) < \infty$ telles que pour toute fonction propre de $\mathbb{T}^d$ de valeur propre suffisamment grande on ait les inégalités* :

$$c(\Sigma)\|\varphi\|_2 \leqslant \|\varphi\|_{L^2(\Sigma)} \leqslant C(\Sigma)\|\varphi\|_2$$

(*où Σ est muni de la mesure de surface*).

Il semble raisonnable de conjecturer que cet énoncé est vrai en toute dimension. Pour la borne supérieure, l'exposant 1/6 dans l'inégalité,

$$\|\varphi\|_{L^2(\Sigma)} \leqslant \lambda^{1/6}\|\varphi\|_2, \quad \text{où} \quad -\Delta\phi = \lambda^2\phi,$$

démontrée dans [1] pour Σ une courbe lisse de courbure positive dans une variété 2-dimensionnelle M (et restant valable en dimension supérieure en supposant Σ une hyper surface lisse à courbure positive) peut être amélioré pour $M = \mathbb{T}^d$:

Théorème 2. *Pour toute dimension d, il existe $\rho(d) < 1/6$ tel que si φ est une fonction propre de $\mathbb{T}^d$, $-\Delta\phi = \lambda^2\phi$ et $\Sigma \subset \mathbb{T}^d$ comme ci-dessus, on ait* :

$$\|\varphi\|_{L^2(\Sigma)} \leqslant c(\Sigma)\lambda^{\rho(d)}\|\varphi\|_2.$$

Le question si $\rho(d) = 0$ pour $d \geqslant 4$ reste ouverte.

La démonstration de ces théorèmes fait intervenir divers ingrédients arithmétiques et analytiques.

1. Introduction and statements

Let M be a smooth Riemannian surface without boundary, Δ the corresponding Laplace–Beltrami operator and Σ a smooth curve in M. Burq, Gérard and Tzvetkov [1] established bounds for the L^2-norm of the restriction of eigenfunctions of Δ to the curve Σ, showing that if $-\Delta\varphi_\lambda = \lambda^2\varphi_\lambda$, $\lambda > 0$, then

$$\|\varphi_\lambda\|_{L^2(\Sigma)} \ll \lambda^{1/4}\|\varphi_\lambda\|_{L^2(M)} \tag{1}$$

and if Σ has non-vanishing geodesic curvature then (1) may be improved to

$$\|\varphi_\lambda\|_{L^2(\Sigma)} \ll \lambda^{1/6}\|\varphi_\lambda\|_{L^2(M)}. \tag{2}$$

Both (1), (2) are saturated for the sphere S^2.

In [1] it is observed that for the flat torus $M = \mathbb{T}^2$, (1) can be improved to

$$\|\varphi_\lambda\|_{L^2(\Sigma)} \ll \lambda^\epsilon\|\varphi_\lambda\|_{L^2(M)}, \quad \forall\epsilon > 0 \tag{3}$$

due to the fact that there is a corresponding bound on the supremum of the eigenfunctions. They raise the question whether in (3) the factor λ^ϵ can be replaced by a constant, that is whether there is a uniform L^2 restriction bound. As pointed out by Sarnak [8], if we take Σ to be a geodesic segment on the torus, this particular problem is essentially equivalent to the currently open question of whether on the circle $|x| = \lambda$, the number of lattice points on an arc of size $\lambda^{1/2}$ admits a uniform bound.

In [1] results similar to (1) are also established in the higher-dimensional case for restrictions of eigenfunctions to smooth submanifolds, in particular (1) holds for codimension-one submanifolds (hypersurfaces) and is sharp for the sphere S^{d-1}. Moreover, (2) remains valid for hypersurfaces with non-vanishing curvature [6].

In this Note we pursue the improvements of (2) for the standard flat d-dimensional tori $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$, considering the restriction to (codimension-one) hypersurfaces Σ with non-vanishing curvature.

Theorem 1.1. *Let $d = 2, 3$ and let $\Sigma \subset \mathbb{T}^d$ be a smooth hypersurface with non-zero curvature. There are constants $0 < c < C < \infty$ and $\Lambda > 0$, all depending on Σ, so that all eigenfunctions φ_λ of the Laplacian on $\mathbb{T}^d$ with $\lambda > \Lambda$ satisfy*:

$$c\|\varphi_\lambda\|_2 \leqslant \|\varphi_\lambda\|_{L^2(\Sigma)} \leqslant C\|\varphi_\lambda\|_2. \tag{4}$$

Observe that for the lower bound, the curvature assumption is necessary, since the eigenfunctions $\varphi(x) = \sin(2\pi n_1 x_1)$ all vanish on the hypersurface $x_1 = 0$. In fact this lower bound implies that a curved hypersurface cannot be contained in the nodal set of eigenfunctions with arbitrarily large eigenvalues.

The proof of Theorem 1.1 (which will be sketched in the next section for the easy case of $d = 2$) permits also to introduce a notion of "relative quantum limit" for restrictions to Σ as above, but we will not discuss this further here.

It is reasonable to believe that Theorem 1.1 holds in any dimension, and one could further conjecture an upper bound without curvature assumptions. At this point, we may only state an improvement of the exponent 1/6:

Theorem 1.2. *For all $d \geqslant 4$ there is $\rho(d) < \frac{1}{6}$ so that if φ_λ is an eigenfunction of the Laplacian on $\mathbb{T}^d$, and $\Sigma \subset \mathbb{T}^d$ is a smooth compact hypersurface with positive curvature, then*

$$\|\varphi_\lambda\|_{L^2(\Sigma)} \ll \lambda^{\rho(d)} \|\varphi\|_2. \tag{5}$$

2. Proof of Theorem 1.1 for $d = 2$

Denote by σ the normalized arc-length measure on the curve Σ. Using the method of stationary phase, one sees that if Σ has non-vanishing curvature then the Fourier transform $\widehat{\sigma}$ decays as

$$\left|\widehat{\sigma}(\xi)\right| \ll |\xi|^{-1/2}, \quad \xi \neq 0. \tag{6}$$

Moreover $|\widehat{\sigma}(\xi)| \leqslant \widehat{\sigma}(0) = 1$ with equality only for $\xi = 0$, hence

$$\sup_{0 \neq \xi \in \mathbb{Z}^2} \left|\widehat{\sigma}(\xi)\right| \leqslant 1 - \delta, \tag{7}$$

for some $\delta = \delta_\Sigma > 0$.

An eigenfunction of the Laplacian on $\mathbb{T}^2$ is a trigonometric polynomial of the form:

$$\varphi(x) = \sum_{|n|=\lambda} \widehat{\varphi}(n) e(n \cdot x) \tag{8}$$

(where $e(z) := e^{2\pi i z}$), all of whose frequencies lie in the set $\mathcal{E} := \mathbb{Z}^2 \cap \lambda S^1$. As is well known, in dimension $d = 2$, $\#\mathcal{E} \ll \lambda^\epsilon$ for all $\epsilon > 0$. Moreover, by a result of Jarnik [7], any arc on λS^1 of length at most $c\lambda^{1/3}$ contains at most two lattice points (Cilleruelo and Cordoba [3] showed that for any $\delta < \frac{1}{2}$, arcs of length λ^δ contain at most $M(\delta)$ lattice points and in [4] it is conjectured that this remains true for any $\delta < 1$). Hence we may partition,

$$\mathcal{E} = \bigsqcup_\alpha \mathcal{E}_\alpha, \tag{9}$$

where $\#\mathcal{E}_\alpha \leqslant 2$ and $\text{dist}(\mathcal{E}_\alpha, \mathcal{E}_\beta) > c\lambda^{1/3}$ for $\alpha \neq \beta$. Correspondingly we may write,

$$\varphi = \sum_\alpha \varphi^\alpha, \quad \varphi^\alpha(x) = \sum_{n \in \mathcal{E}_\alpha} \widehat{\varphi}(n) e(nx), \tag{10}$$

so that $\|\varphi\|_2^2 = \sum_\alpha \|\varphi^\alpha\|_2^2$, and

$$\int_\Sigma |\varphi|^2 \, d\sigma = \sum_\alpha \sum_\beta \int_\Sigma \varphi^\alpha \overline{\varphi^\beta} \, d\sigma. \tag{11}$$

Applying (6) we see that $\int_\Sigma \varphi^\alpha \overline{\varphi^\beta} \, d\sigma \ll \lambda^{-1/6}$ if $\alpha \neq \beta$ and because $\#\mathcal{E} \ll \lambda^\epsilon$ the total sum of these non-diagonal terms is bounded by $\lambda^{-1/6+\epsilon} \|\varphi\|_2^2$. It suffices then to show that the diagonal terms satisfy

$$\delta \|\phi^\alpha\|_2^2 \leqslant \int_\Sigma |\phi^\alpha|^2 \, d\sigma \leqslant 2 \|\phi^\alpha\|_2^2. \tag{12}$$

This is clear if $\mathcal{E}_\alpha = \{n\}$, while if $\mathcal{E}_\alpha = \{m, n\}$, then

$$\int_\Sigma |\phi^\alpha|^2 \, d\sigma = \left|\widehat{\varphi}(m)\right|^2 + \left|\widehat{\varphi}(n)\right|^2 + 2 \, \text{Re} \, \widehat{\varphi}(m) \overline{\widehat{\varphi}(n)} \widehat{\sigma}(m - n), \tag{13}$$

and then (12) follows from (7). Thus we get Theorem 1.1 for $d = 2$.

3. The higher-dimensional case

The proof of Theorem 1.1 for dimension $d = 3$ is considerably more involved. Arguing along the lines of the two-dimensional case gives an upper bound of λ^ϵ. To get the uniform bound of Theorem 1.1 for $d = 3$ and the results of Theorem 1.2, we need to replace the upper bound (6) for the Fourier transform of the hypersurface measure by an asymptotic expansion, and then exploit cancellation in the resulting exponential sums over the sphere. A key ingredient there is controlling the number of lattice points in spherical caps.

3.1. Distribution of lattice points on spheres

To state some relevant results, denote as before by $\mathcal{E} = \mathbb{Z}^d \cap \lambda S^{d-1}$ the set of lattice points on the sphere of radius λ. We have $\#\mathcal{E} \ll \lambda^{d-2+\epsilon}$. Let $F_d(\lambda, r)$ be the maximal number of lattice points in the intersection of $\mathcal{E}$ with a spherical cap of size $r > 1$. A higher-dimensional analogue of Jarnik's theorem implies that if $r \ll \lambda^{1/(d+1)}$ then all lattice points in such a cap are co-planar, hence $F_d(r, \lambda) \ll r^{d-3+\epsilon}$ in that case, for any $\epsilon > 0$. For larger caps, we show:

Proposition 3.1.

(i) *Let $d = 3$. Then for any $\eta < \frac{1}{15}$,*

$$F_3(\lambda, r) \ll \lambda^\epsilon \left(r \left(\frac{r}{\lambda} \right)^\eta + 1 \right). \tag{14}$$

(ii) *Let $d = 4$. Then*

$$F_4(\lambda, r) \ll \lambda^\epsilon \left(\frac{r^3}{\lambda} + r^{3/2} \right). \tag{15}$$

(iii) *For $d \geqslant 5$ we have*

$$F_d(\lambda, r) \ll \lambda^\epsilon \left(\frac{r^{d-1}}{\lambda} + r^{d-3} \right) \tag{16}$$

(the factor λ^ϵ is redundant for large d).

The term r^{d-1}/λ concerns the equidistribution of $\mathcal{E}$, while the term r^{d-3} measures deviations related to accumulation in lower-dimensional strata.

The second result expresses a mean-equidistribution property of $\mathcal{E}$. Partition the sphere λS^2 into sets C_α of size $\lambda^{1/2}$, for instance by intersecting with cubes of that size. Since $\#\mathcal{E} \ll \lambda^{1+\epsilon}$, one may expect that $\#C_\alpha \cap \mathcal{E} \ll \lambda^\epsilon$. Using Siegel's mass formula for the number of representations of an integral quadratic form by the genus of a quadratic form, we show (in a joint work with P. Sarnak [2]) that this holds in the mean square:

Proposition 3.2.

$$\sum_\alpha \left[\#(\mathcal{E} \cap C_\alpha) \right]^2 \ll \lambda^{1+\epsilon}, \quad \forall \epsilon > 0. \tag{17}$$

3.2. Exponential sums on the sphere

Let $1 < r < \lambda$ and let C, C' be spherical r-caps on λS^{d-1} of mutual distance at least $10r$. Following the argument for $d = 2$, we need to bound exponential sums of the form

$$\sum_{n \in C} \sum_{n' \in C'} \widehat{\varphi}(n) \overline{\widehat{\varphi}(n')} e\left(h(n - n') \right), \quad \|\varphi\|_2 = 1 \tag{18}$$

where h is the support function of the hyper-surface Σ, which appears in the asymptotic expansion of the Fourier transform of the surface measure on Σ, see [5]. For instance, in the case that $\Sigma = \{|x| = 1\}$ is the unit sphere then $h(\xi) = |\xi|$.

Consider from now on the case $d = 3$. For $r < \lambda^{1-\epsilon}$ we simply estimate (18) by $F_3(\lambda, r)$ (see (14)). When $\lambda^{1-\epsilon} < r < \lambda$ this bound does not suffices and we need to exploit cancellation in the sum (18).

Proposition 3.3. *There is $\delta > 0$ so that* (18) *admits a bound of $\lambda^{1-\delta}$ for $\lambda \gg 1$.*

This statement depends essentially on the equidistribution of $\mathcal{E}$ in $\sqrt{\lambda}$-caps, as expressed in Proposition 3.2.

We finally formulate an example of a bilinear estimate involved in analyzing (18).

Proposition 3.4. *Let $\beta \gg 1$ and $X, Y \subset [0, 1]$ arbitrary discrete sets such that $|x - x'|, |y - y'| > \beta^{-1/2}$ for $x \neq x' \in X$ and $y \neq y' \in Y$. Then*

$$\left| \sum_{x \in X} \sum_{y \in Y} e(\beta x y + \beta^{1/3} x^2 y^2) \right| \ll \beta^{23/24+\epsilon}, \tag{19}$$

for all $\epsilon > 0$.

Note that the non-linear term in the phase function is crucial for a non-trivial bound to hold in this setting.

References

[1] N. Burq, P. Gérard, N. Tzvetkov, Restrictions of the Laplace–Beltrami eigenfunctions to submanifolds, Duke Math. J. 138 (3) (2007) 445–486.
[2] J. Bourgain, Z. Rudnick, P. Sarnak, in preparation.
[3] J. Cilleruelo, A. Córdoba, Trigonometric polynomials and lattice points, Proc. Amer. Math. Soc. 115 (4) (1992) 899–905.
[4] J. Cilleruelo, A. Granville, Lattice points on circles, squares in arithmetic progressions and sumsets of squares, in: Additive Combinatorics, in: CRM Proc. Lecture Notes, vol. 43, Amer. Math. Soc., Providence, RI, 2007, pp. 241–262.
[5] C.S. Herz, Fourier transforms related to convex sets, Ann. of Math. (2) 75 (1962) 81–92.
[6] R. Hu, L^p norm estimates of eigenfunctions restricted to submanifolds, Forum Math., in press.
[7] V. Jarnik, Über die Gitterpunkte auf konvexen Kurven, Math. Z. 24 (1) (1926) 500–518.
[8] P. Sarnak, Letter to Reznikov, see http://www.math.princeton.edu/sarnak/, June 2008.