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COMPTES RENDUS MATHEMATIQUE

Partial Differential Equations/Numerical Analysis

Asymptotic expansions of the eigenvalues of a 2-D boundary-value problem relative to two cavities linked by a hole of small size ☆

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Abstract

This Note presents the derivation of the 2nd-order asymptotic expansion of the eigenvalues and the eigenfunctions of the operator associated to an interior elliptic equation supplemented by a Dirichlet boundary condition on a domain consisting of two cavities linked by a hole of small size. The asymptotic expansion is carried out with respect to the size of the hole. The main feature of the method is to yield a robust numerical procedure making it possible to compute the eigenvalues without resorting to a refined mesh around the hole. ***To cite this article: A. Bendali et al., C. R. Acad. Sci. Paris, Ser. I 347 (2009).***

Résumé

Développement asymptotique des valeurs et fonctions propres d'un problème aux limites 2-D relatif à deux cavités reliées par un trou de petite taille.[1] Cette Note présente la dérivation du développement asymptotique au 2nd ordre des valeurs et des fonctions propres de l'opérateur associé à une équation elliptique complétée par une condition aux limites de Dirichlet sur un domaine formé de deux cavités reliées par un trou de petite taille. Le développement asymptotique est effectué relativement à la taille du trou. La principale caractéristique de la méthode est de donner lieu à une procédure numérique permettant de calculer les valeurs propres sans recourir à un maillage raffiné autour du trou. ***Pour citer cet article : A. Bendali et al., C. R. Acad. Sci. Paris, Ser. I 347 (2009).***

Version française abrégée

Dans cette Note, nous présentons des résultats relatifs aux valeurs propres (3) d'un problème aux limites elliptiques avec données au bord de Dirichlet posé sur un domaine constitué de deux cavités reliées par un trou de taille δ (Fig. 1).

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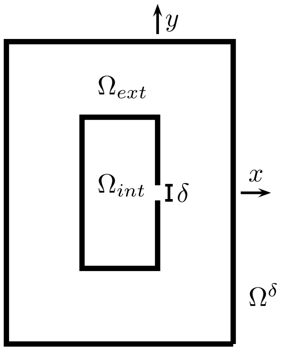

Fig. 1. The domain Ω^δ.

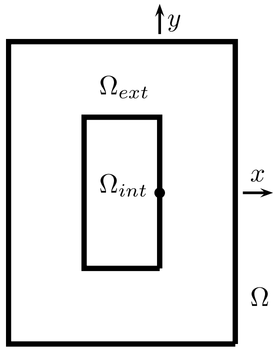

Fig. 2. The domain Ω.

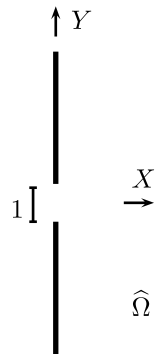

Fig. 3. The domain $\widehat{\Omega}$.

Nous montrons que lorsque $\delta \to 0$, la $n^{\text{ème}}$ valeur propre λ_n^δ du problème (3) tend vers la $n^{\text{ème}}$ valeur propre λ_n du problème limite (4). Plus précisément, en utilisant la technique des développements asymptotiques raccordés, nous obtenons le développement asymptotique à l'ordre deux des λ_n^δ pour $\delta \to 0$.

Théorème. *Si la valeur propre λ_n du problème limite est simple, alors λ_n^δ admet le développement* (13) *avec $\lambda_{n,2}$ défini par* (11) *et u_n la fonction propre associée à λ_n* (*voir* (4)).

Ce développement asymptotique peut être utilisé pour calculer une approximation numérique ne nécessitant ni prise en compte du trou dans la géométrie ni raffinement de maillage au voisinage du trou. Nous terminons cette Note en présentant quelques simulations numériques. Ces expériences valident les conclusions théoriques comme cela est illustré par les Figs. 4–7 et 9.

1. A 2D eigenvalue problem

Let Ω_{int} and Ω_{ext} be two open subsets of $\mathbb{R}^2$ such that

$$\Omega_{\text{int}} \cap \Omega_{\text{ext}} = \emptyset \quad \text{and} \quad \exists \delta_0 > 0\colon \big(\{0\} \times]-\delta_0; \delta_0[\big) \in \partial\Omega_{\text{int}} \cap \partial\Omega_{\text{ext}}. \tag{1}$$

For $\delta < \delta_0$, we consider the domain Ω^δ consisting of Ω_{ext} and Ω_{int} linked by a hole of width δ

$$\Omega^\delta := \Omega_{\text{int}} \cup \Omega_{\text{ext}} \cup \left(\{0\} \times \left]-\frac{\delta}{2}; \frac{\delta}{2}\right[\right) \subset \mathbb{R}^2 \tag{2}$$

which tends to $\Omega := \Omega_{\text{int}} \cup \Omega_{\text{ext}} \subset \mathbb{R}^2$ when $\delta \to 0$ (see Figs. 1 and 2).

With respect to these domains, we consider the following eigenvalue problems:

$$\begin{cases} u_n^\delta : \Omega^\delta \to \mathbb{R}, & u_n^\delta \neq 0,\ \lambda_n^\delta \in \mathbb{R}, \\ -\nabla \cdot (a\nabla u_n^\delta) = \lambda_n^\delta b u_n^\delta & \text{in } \Omega^\delta, \\ u_n^\delta = 0 & \text{on } \partial\Omega^\delta, \end{cases} \tag{3}$$

$$\begin{cases} u_n : \Omega \to \mathbb{R}, & u_n \neq 0,\ \lambda_n \in \mathbb{R}, \\ -\nabla \cdot (a\nabla u_n) = \lambda_n b u_n & \text{in } \Omega, \\ u_n = 0 & \text{on } \partial\Omega, \end{cases} \tag{4}$$

with $a : \Omega \to \mathbb{R}$ and $b : \Omega \to \mathbb{R}$ two sides bounded functions of Ω that can be expanded in neighborhood of $(0,0)$ with the form

$$a|_{\Omega_{\text{ext}}}(x,y) = \sum_{i,j \geqslant 0} a_{i,j}^{\text{ext}} x^i y^j, \qquad b|_{\Omega_{\text{ext}}}(x,y) = \sum_{i,j \geqslant 0} b_{i,j}^{\text{ext}} x^i y^j \quad \text{with } a_{i,j}^{\text{ext}}, b_{i,j}^{\text{ext}} \in \mathbb{R}, \tag{5}$$

$$a|_{\Omega_{\text{int}}}(x,y) = \sum_{i,j \geqslant 0} a_{i,j}^{\text{int}} x^i y^j, \qquad b|_{\Omega_{\text{int}}}(x,y) = \sum_{i,j \geqslant 0} b_{i,j}^{\text{int}} x^i y^j \quad \text{with } a_{i,j}^{\text{int}}, b_{i,j}^{\text{int}} \in \mathbb{R} \tag{6}$$

and satisfy $\inf_{(x,y)\in\Omega} a(x,y) > 0$ and $\inf_{(x,y)\in\Omega} b(x,y) > 0$.

The respective eigenfunctions u_n^δ and u_n can be chosen so that they constitute a bi-orthogonal basis of L^2 and H^1 and are numbered so that

$$\lambda_0^\delta \leqslant \lambda_1^\delta \leqslant \cdots, \quad \lim_{n\to+\infty} \lambda_n^\delta = +\infty, \tag{7}$$

$$\lambda_0 \leqslant \lambda_1 \leqslant \cdots, \quad \lim_{n\to+\infty} \lambda_n = +\infty. \tag{8}$$

When a and b are constant functions, the 2nd-order asymptotic expansion of λ_n^δ can be derived from an adaptation of the results on scattering poles in [5]. Our purpose in this Note is to generalize this result for non-constant and non-continuous across the hole a and b and to show that this asymptotic expansion defines a numerical procedure yielding an accurate approximation of λ_n^δ for small δ which requires no mesh refinement. For simplicity, we furthermore assume that eigenvalues $(\lambda_n)_{n\geqslant 0}$ are simple, that is,

$$\lambda_n = \lambda_p \quad \text{if and only if } p = n. \tag{9}$$

Remark 1. Condition (9) is not crucial. It is mostly considered to avoid unnecessary complications due to resonance phenomena between two close eigenvalues of the elliptic operator in Ω^δ. Moreover, it implies that all the eigenvectors related to Ω are eigenvectors relatively to either Ω_{int} or of Ω_{ext}. As a result, one has $u_n = 0$ in Ω_{int} or in Ω_{ext}.

To end this introduction, we point out that the problem of a wall perforated by a small hole has been previously widely studied both from the theoretical and the numerical point of view (see [10,9,11] for a comprehensive bibliography). It is also worth mentioning that this problem is similar to the Dumbbell problem also called Helmholtz resonator (the eigenvalue problem related to two cavities linked by a thin slot of finite length (see [1,2,4]).

2. The asymptotic expansions

The 2nd-order asymptotic expansions of λ_n^δ and u_n^δ are dealt with simultaneously. The eigenfunctions u_n^δ are characterized by a boundary layer behavior in the vicinity of the small hole. Therefore, we look for two asymptotic expansions of u_n^δ. The first one, yielding the overall behavior of u_n^δ outside the boundary layer, is expressed by means of the slow variable (x, y) and is called the far-field expansion. The second one is the near-field expansion and is written using the fast variable $(X, Y) = (x/\delta, y/\delta)$, and is defined on $\widehat{\Omega} := \mathbb{R}^2 \setminus (\{0\} \times (]-\infty, -\frac{1}{2}[\cup], \frac{1}{2}, +\infty[))$, an infinite domain describing a normalized shape of the hole (see Fig. 3). The near-field expansion is used to approximate u_n^δ in a small neighborhood of the hole. Since both are two approximations of the same function u_n^δ, they have to satisfy some matching conditions in some intermediate zone. This approach, often called Matched Asymptotic Expansions (MAE), was widely studied and now became rather well understood (see [6,12] and the references therein). This technique is often considered as a formal reasoning. However, it can become a rigorous approach if error estimates can be established (see, for example, [7,8]). The 2nd-order asymptotic expansions are respectively given by

$$\begin{aligned} &\lambda_n^\delta \approx \lambda_n + \delta^2 \lambda_{n,2}, \\ &u_n^\delta(x, y) \approx u_n(x, y) + \delta^2 u_{n,2}(x, y), \\ &u_n^\delta(\delta X, \delta Y) = \Pi^\delta(X, Y) \approx \delta \Pi_{n,1}(X, Y) + \delta^2 \Pi_{n,2}(X, Y), \end{aligned} \tag{10}$$

with

$$\lambda_{n,2} = \begin{cases} -\frac{\pi}{8} \frac{(a_{0,0}^{\text{int}})^2}{a_{0,0}^{\text{int}} + a_{0,0}^{\text{ext}}} \frac{(\partial_x u_n|_{\Omega_{\text{int}}}(0,0))^2}{\int_{\Omega_{\text{int}}} b|u_n|^2} & \text{if } u_n = 0 \text{ in } \Omega_{\text{ext}}, \\ -\frac{\pi}{8} \frac{(a_{0,0}^{\text{ext}})^2}{a_{0,0}^{\text{ext}} + a_{0,0}^{\text{int}}} \frac{(\partial_x u_n|_{\Omega_{\text{ext}}}(0,0))^2}{\int_{\Omega_{\text{ext}}} b|u_n|^2} & \text{if } u_n = 0 \text{ in } \Omega_{\text{int}}, \end{cases} \tag{11}$$

$$\begin{cases} u_{n,2} : \Omega \to \mathbb{R}, \\ \nabla \cdot (a \nabla u_{n,2}) + \lambda_n b u_{n,2} = -\lambda_{n,2} b u_n & \text{in } \Omega, \\ u_{n,2} = 0 & \text{on } \partial\Omega \setminus \{\mathbf{0}\}, \\ u_{n,2} - (a_{0,0}^{\text{int}} \partial_x u_n|_{\Omega_{\text{int}}}(0,0) - a_{0,0}^{\text{ext}} \partial_x u_n|_{\Omega_{\text{ext}}}(0,0)) \frac{1}{8} \frac{1}{a_{0,0}^{\text{ext}} + a_{0,0}^{\text{int}}} \frac{x}{x^2+y^2} \underset{\|(x,y)\|\to 0}{=} O(1) & \text{in } \Omega_{\text{int}}, \\ u_{n,2} + (a_{0,0}^{\text{int}} \partial_x u_n|_{\Omega_{\text{int}}}(0,0) - a_{0,0}^{\text{ext}} \partial_x u_n|_{\Omega_{\text{ext}}}(0,0)) \frac{1}{8} \frac{1}{a_{0,0}^{\text{int}} + a_{0,0}^{\text{ext}}} \frac{x}{x^2+y^2} \underset{\|(x,y)\|\to 0}{=} O(1) & \text{in } \Omega_{\text{ext}}. \end{cases} \tag{12}$$

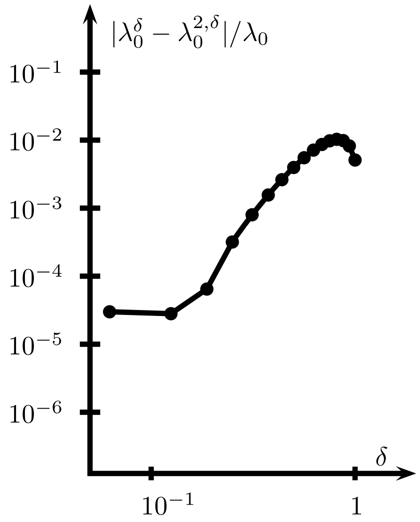

Fig. 4. Relative error $|\lambda_0^\delta - \lambda_0^{2,\delta}|/\lambda_0$ for Ω_0^δ.

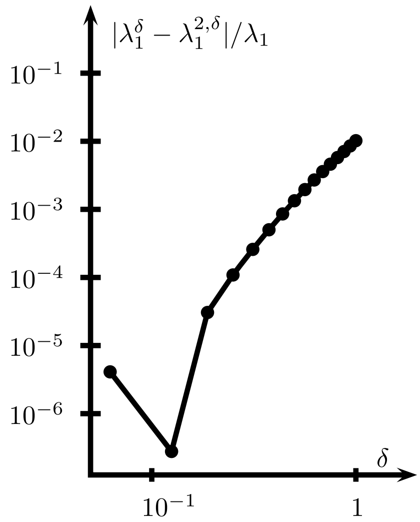

Fig. 5. Relative error $|\lambda_1^\delta - \lambda_1^{2,\delta}|/\lambda_1$ for Ω_1^δ.

$$\begin{cases} \Pi_{n,1} : \widehat{\Omega} \to \mathbb{R}, & \\ -\nabla \cdot (a_{0,0} \nabla \Pi_{n,1}) = 0 & \text{in } \widehat{\Omega}, \\ \Pi_{n,1} = 0 & \text{on } \partial\widehat{\Omega}, \\ \Pi_{n,1} - X \partial_x u_n|_{\Omega_{\text{int}}}(0,0) = \underset{\|(X,Y)\| \to +\infty}{O(1)} & \text{if } X > 0, \\ \Pi_{n,1} - X \partial_x u_n|_{\Omega_{\text{ext}}}(0,0) = \underset{\|(X,Y)\| \to +\infty}{O(1)} & \text{if } X < 0, \end{cases}$$

$$\begin{cases} \Pi_{n,2} : \widehat{\Omega} \to \mathbb{R}, & \\ -\nabla \cdot (a_{0,0} \nabla \Pi_{n,2}) = \nabla \cdot ((a_{1,0} X + a_{0,1} Y) \nabla \Pi_{n,1}) & \text{in } \widehat{\Omega}, \\ \Pi_{n,2} = 0 & \text{on } \partial\widehat{\Omega}, \\ \Pi_{n,2} - XY \partial^2_{xy} u_n|_{\Omega_{\text{int}}}(0,0) + \frac{X^2}{2} \frac{a_{1,0}^{\text{int}}}{a_{0,0}^{\text{int}}} \partial_x u_n|_{\Omega_{\text{int}}}(0,0) = \underset{\|(X,Y)\| \to +\infty}{O(1)} & \text{if } X > 0, \\ \Pi_{n,2} - XY \partial^2_{xy} u_n|_{\Omega_{\text{ext}}}(0,0) + \frac{X^2}{2} \frac{a_{1,0}^{\text{ext}}}{a_{0,0}^{\text{ext}}} \partial_x u_n|_{\Omega_{\text{ext}}}(0,0) = \underset{\|(X,Y)\| \to +\infty}{O(1)} & \text{if } X < 0, \end{cases}$$

with $a_{i,j}$ the piecewise constant function satisfying $a_{i,j}|_{\Omega_{\text{int}}} = a_{i,j}^{\text{int}}$ and $a_{i,j}|_{\Omega_{\text{ext}}} = a_{i,j}^{\text{ext}}$.

Theorem. *Under hypothesis* (9)*, the eigenvalue* λ_n^δ *is given by*

$$\lambda_n^\delta = \lambda_n + \delta^2 \lambda_{n,2} + \underset{\delta \to 0}{O}\big(\delta^3 \ln \delta\big). \tag{13}$$

Proof. Mimicking the proof given in [3], this theorem can be obtained. □

Remark 2. For a small δ, formula (13) provides a way to compute an approximation of λ_n^δ involving only the computation of the eigenmodes u_n of the elliptic operator in Ω. As a result, even for small δ, no mesh refinement is necessary to compute accurate approximations of the eigenvalues relative to Ω^δ.

3. Numerical experiments

In this section, we compare an accurate approximation of λ_n^δ obtained through a direct numerical simulation based on a very refined mesh to its 2nd-order expansion $\lambda_n^{2,\delta} = \lambda_n + \delta^2 \lambda_{n,2}$ for the domain Ω^δ (see Fig. 8). We consider

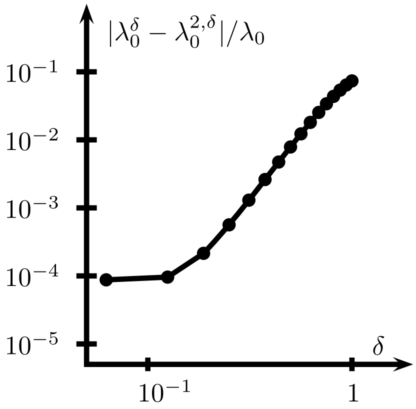

Fig. 6. Relative error $|\lambda_0^\delta - \lambda_0^{2,\delta}|/\lambda_0$ for Ω_0^δ.

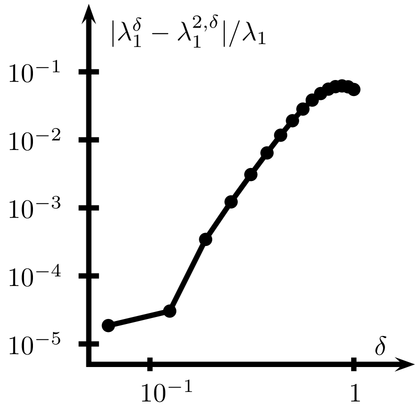

Fig. 7. Relative error $|\lambda_1^\delta - \lambda_1^{2,\delta}|/\lambda_1$ for Ω_1^δ.

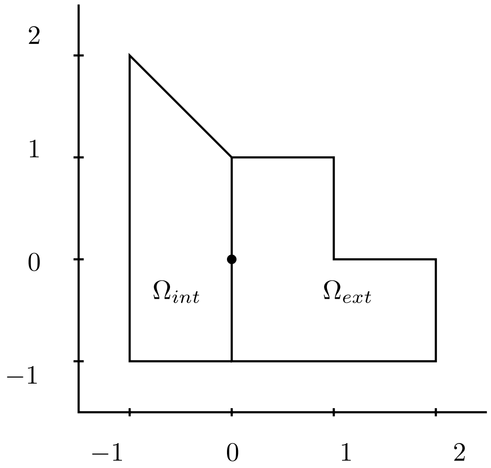

Fig. 8. The domain Ω.

	1st experiment		2nd experiment	
n	0	1	0	1
λ_n	11.655	16.919	6.656	10.46
$\lambda_{n,2}$	13.857	20.948	11.425	17.252

Fig. 9. The eigenvalues coefficients.

two different sets of data for a and b. In the first experiment the coefficients a and b are discontinuous through the hole

$$a|_{\Omega_{\text{ext}}}(x, y) = 2, \qquad a|_{\Omega_{\text{int}}}(x, y) = 1, \qquad b|_{\Omega_{\text{ext}}}(x, y) = \frac{1}{2}, \quad \text{and} \quad b|_{\Omega_{\text{int}}}(x, y) = 1. \tag{14}$$

In the second experiment, varying coefficients are considered

$$a(x, y) = 2 - \sin(x)\cos(x) \quad \text{and} \quad b(x, y) = 3 + \cos(x + y). \tag{15}$$

Using the finite element library GETFEM (see http://home.gna.org/getfem/), we compute a P^3-continuous finite element approximation of u_n^δ and λ_n^δ on a uniform refined triangular mesh ($h = 0.03125$) and a P^6-continuous finite element approximation of u_n^δ and λ_n^δ on a non-refined triangular mesh ($h = 0.5$). The results of the numerical experiments are reported in Figs. 4–7 and 9. For practical reasons, mainly due to limitations in memory storage, only holes of size $\delta \geqslant 0.0625$ were considered. The results, show an excellent agreement with those predicted by the theory.

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