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Harmonic Analysis/Functional Analysis

Finite square function implies integer dimension

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Abstract

Following a recent paper (Tolsa and Ruiz de Villa, 2008) we show that the finiteness of square function associated with the Riesz transforms with respect to Hausdorff measure H^s implies that s is integer. ***To cite this article: S. Mayboroda, A. Volberg, C. R. Acad. Sci. Paris, Ser. I 347 (2009).***

Résumé

Pour que la fonction carrée de mesure de Hausdorff soit finie il faut que la dimension de mesure soit entier. On peut modifier l'article recent (Tolsa and Ruiz de Villa, 2008) pour démontrer que la convergence de la fonction carrée associée aux transformations de Riesz de mesure de Hausdorff H^s implique que s est un nombre entier. ***Pour citer cet article : S. Mayboroda, A. Volberg, C. R. Acad. Sci. Paris, Ser. I 347 (2009).***

Version française abrégée

Soit E un compact de $\mathbb{R}^m$ tel que $0 < H^s(E) < \infty$, $0 < s \leqslant m$.

Les s-densités supérieure et inférieure d'une mesure de Borel μ en un point x sont respectivement données par

$$\theta_\mu^{s,*}(x) := \limsup_{r\to 0} \frac{\mu(B(x,r))}{r^s} \quad \text{et} \quad \theta_{\mu,*}^{s}(x) := \liminf_{r\to 0} \frac{\mu(B(x,r))}{r^s}, \tag{1}$$

où $B(x,r)$ est une boule de rayon $r > 0$ centrée en $x \in \mathbb{R}^m$. Le théorème de Marstrand énonce que, si la densité $\theta_\mu^s(x) = \theta_\mu^{s,*}(x) = \theta_{\mu,*}^s(x)$ de $\mu := H^s|E$ existe presque partout par rapport à μ, alors s est un nombre entier. Un problème bien connu et non encore réglé d'analyse non-homogène dans $\mathbb{R}^m$ est d'aboutir à la même conclusion à partir de l'hypothèse que ($\mu := H^s|E$)

$$\sup_{\varepsilon>0} \left| R_\varepsilon^s \mu(x)\right| < \infty, \quad \mu\text{-p.p. } x \in E. \tag{2}$$

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Ici, pour une mesure de Borel μ dans $\mathbb{R}^m$ et un nombre $s \in (0, m]$, la transformation de Riesz de μ est définie par

$$R^s\mu(x) := \int \frac{x-y}{|x-y|^{s+1}}\,\mathrm{d}\mu(y), \quad x \notin \operatorname{supp}\mu, \tag{3}$$

et la transformation de Riesz tronquée est

$$R^s_\varepsilon\mu(x) := \int\limits_{|x-y|>\varepsilon} \frac{x-y}{|x-y|^{s+1}}\,\mathrm{d}\mu(y), \quad x \in \mathbb{R}^m,\ \varepsilon > 0. \tag{4}$$

Si on remplace l'hypothèse (2) par le fait que l'opérateur de Riesz est borné dans $L^2(\mu)$:

$$\sup_{\varepsilon>0} \left\| R^s_\varepsilon : L^2(\mu) \to L^2(\mu) \right\| < \infty \tag{5}$$

on arrive à un problème pratiquement équivalent. Cette question est connue comme un problème de Guy David.

Selon la théorie de David et Semmes, si on remplace un opérateur de Riesz en (5) par tous les opérateurs de Calderón–Zygmund d'une certaine classe, on obtient la rectifibialité de E (s est entier et E contient « les gros morceaux de graphes lipschitziens »).

Supposons que seuls les opérateurs de Calderón–Zygmund obtenus par la procédure de Littlewood–Paley soient bornés dans $L^2(\mu)$. Il s'agit d'une classe beaucoup plus petite que la classe de David–Semmes. On démontre ici que cette classe effectivement est suffisante pour conclure que s est un nombre entier. Autrement dit, nous prouvons qu'il est suffisant d'avoir un seul opérateur borné dans $L^2(\mu)$, et que cet opérateur est la fonction carrée de type de Lusin, qui, entre parenthèses, est la moyenne des opérateurs de Littlewood–Paley.

Malheureusement, on ne sait pas comment passer de (5) au fait que la fonction carrée soit bornée, ou aux faits que les opérateur de Littlewood–Paley soient bornés.

Le principal résultat de cette communication est le suivant :

Théorème 0.1. *Soit μ une mesure de Radon finie sur $\mathbb{R}^m$ avec*

$$0 < \theta^{s,*}_\mu(x) < \infty, \quad \mu\text{-p.p. } x \in \mathbb{R}^m. \tag{6}$$

Soit $s \in (0, m]$, supposons que

$$S^s\mu(x) := \left(\int\limits_0^1 \left| R^s_{u,2u}\mu(x) \right|^2 \frac{\mathrm{d}u}{u} \right)^{1/2} < \infty, \quad \mu\text{-p.p. } x \in \mathbb{R}^m. \tag{7}$$

Alors, s est un nombre entier.

Soit $E \subset \mathbb{R}^m$ un compact, alors pour $\mu = H^s|E$, l'hypothèse (6) est automatique.

Pour conclure le fait que s est un nombre entier à partir de l'hypothèse (2) ou à partir de l'hypothèse (5) il suffit de démontrer :

La conjecture. *Il existe une partie $E' \subset E = \operatorname{supp}\mu$, $\mu E' > 0$ telle que*

$$\lim_{\varepsilon\to 0} R_{\varepsilon,2\varepsilon}\mu(x) = 0, \quad \mu\text{-p.p. } x \in E'. \tag{8}$$

Il semble que (8) est quasi évident, si on a (5), par exemple. La première étape peut-être est de démontrer :

$$\lim_{\varepsilon\to 0} \int\limits_{E'} \left| R_{\varepsilon,2\varepsilon}\mu(x) \right|^2 \mathrm{d}\mu(x) = 0. \tag{9}$$

Mais cette simplicité n'est qu'apparente. On a besoin de nouvelles idées pour démontrer la conjecture ou même (9).

1. Introduction

For a Borel measure μ in $\mathbb{R}^m$ and $s \in (0, m]$ the s-Riesz transform of μ is defined as

$$R^s\mu(x) := \int \frac{x-y}{|x-y|^{s+1}}\,\mathrm{d}\mu(y), \quad x \notin \operatorname{supp}\mu, \tag{10}$$

and the truncated Riesz transform is given by

$$R^s_\varepsilon\mu(x) := \int\limits_{|x-y|>\varepsilon} \frac{x-y}{|x-y|^{s+1}}\,\mathrm{d}\mu(y), \qquad R^s_{\varepsilon,\eta}\mu(x) := R^s_\eta\mu(x) - R^s_\varepsilon\mu(x), \tag{11}$$

where $x \in \mathbb{R}^m$, $\eta > \varepsilon > 0$.

Further, recall that the upper and lower s-dimensional densities of μ at x are given by

$$\theta^{s,*}_\mu(x) := \limsup_{r\to 0} \frac{\mu(B(x,r))}{r^s} \quad \text{and} \quad \theta^s_{\mu,*}(x) := \liminf_{r\to 0} \frac{\mu(B(x,r))}{r^s}, \tag{12}$$

respectively, where $B(x,r)$ is the ball of radius $r > 0$ centered at $x \in \mathbb{R}^m$.

It has been proved in [6] and [7] that whenever $0 \leqslant s \leqslant 1$ and μ is a finite Radon measure with $0 < \theta^{s,*}_\mu(x) < \infty$, for μ-a.e. $x \in \mathbb{R}^m$, the condition

$$\sup_{\varepsilon>0} \left|R^s_\varepsilon\mu(x)\right| < \infty, \quad \mu\text{-a.e. } x \in \mathbb{R}^m, \tag{13}$$

implies that $s \in \mathbb{Z}$. Moreover, an analogous result has been obtained in [11] for all $0 \leqslant s \leqslant m$ under a stronger assumption that $0 < \theta^s_{\mu,*}(x) \leqslant \theta^{s,*}_\mu(x) < \infty$. However, neither the curvature methods of [6,7], nor the tangent measure techniques in [11] could be applied to establish that (13) implies $s \in \mathbb{Z}$ for all $0 \leqslant s \leqslant m$ assuming only $0 < \theta^{s,*}_\mu(x) < \infty$.

In [10] the authors proved that the latter statement holds if the condition (13) is substituted by the existence of the principal value $\lim_{\varepsilon\to 0} R^s_\varepsilon\mu(x)$, μ-a.e. $x \in \mathbb{R}^m$. In the present work we refine the techniques of [10] and establish the following result:

Theorem 1.1. *Let μ be a finite Radon measure in $\mathbb{R}^m$ with*

$$0 < \theta^{s,*}_\mu(x) < \infty \quad \text{for } \mu\text{-a.e. } x \in \mathbb{R}^m. \tag{14}$$

Furthermore, assume that for some $s \in (0, m]$ the square function

$$S^s\mu(x) := \left(\int\limits_0^\infty \left|R^s_{t,2t}\mu(x)\right|^2 \frac{\mathrm{d}t}{t}\right)^{1/2} < \infty, \quad \mu\text{-a.e. } x \in \mathbb{R}^m. \tag{15}$$

Then $s \in \mathbb{Z}$.

In fact, we also prove the following closely related result which is a strengthening of the main results in [10]:

Theorem 1.2. *Let μ be a finite Radon measure in $\mathbb{R}^m$ satisfying* (14)*. Furthermore, assume that for some $s \in (0, m]$ we have*

$$\lim_{\varepsilon\to 0} R^s_{\varepsilon,2\varepsilon}\mu(x) = 0, \quad \mu\text{-a.e. } x \in \mathbb{R}^m. \tag{16}$$

Then $s \in \mathbb{Z}$.

This circle of problems goes back, in particular, to the work of David and Semmes [1,2], where the authors showed, under certain assumptions on the measure μ, that the L^2 boundedness of a large class of singular integral operators implies that s is an integer and μ is uniformly rectifiable, that is, the support of μ contains “large pieces of Lipschitz graphs” – see [1,2] for details. The ultimate goal, which seems to be out of reach at the moment, is to prove that a similar conclusion holds purely under the assumption that the Riesz transform is bounded in L^2, i.e., that the

Riesz transform alone encodes the geometric information about the underlying measure. The achievements in [5,12,4] showed that the L^2-boundedness of the Riesz transform, suitably interpreted, is almost equivalent to the condition (13). However, under the assumption (13) the problem seems to be just as challenging. In both cases the question has only been resolved for $s = 1$ [3,8,9], by the methods involving curvature of measures.

In this vein, we would like to point out that by Khinchin's inequality (15) can be viewed *almost* as a condition

$$\mathbb{E}\left|\sum_{k\in\mathbb{Z}} \varepsilon_k R^s_{2^{-k},2^{-k+1}}\mu(x)\right| < \infty, \quad \mu\text{-a.e. } x \in \mathbb{R}^m, \tag{17}$$

where ε_k are independent random variables taking the values -1 and 1 with probability $1/2$ each. Therefore, in order to guarantee $s \in \mathbb{Z}$, it is sufficient to assume only that the singular integrals of the type $\sum_{k=0}^{\infty} \varepsilon_k R^s_{2^{-k},2^{-k+1}}\mu(x)$ are uniformly bounded.

Finally, the s-dimensional Hausdorff measure H^s of a set E with $0 < H^s(E) < \infty$ satisfies the condition (14), and hence, the results of Theorems 1.1 and 1.2 remain valid in this context.

2. Preliminary estimates

Our proof largely relies on the estimates for a slightly modified version of the Riesz transform that were obtained in [10]. To be precise, let us consider the operator

$$R^{s,\varphi}_\varepsilon \mu(x) := \int \varphi\left(\frac{|x-y|^2}{\varepsilon^2}\right) \frac{x-y}{|x-y|^{s+1}}\,\mathrm{d}\mu(y), \quad \varepsilon > 0, \tag{18}$$

where $\varphi = \varphi_\rho$ is a C^2 function depending on the parameter $\rho \in (0, 1/2)$, to be determined below, with $\operatorname{supp}\varphi \subset [0, 1 + \rho + 2\rho^2]$ and such that

(i) $\varphi(r) = r^{\frac{s+1}{2}}$ for $0 \leqslant r \leqslant 1$, $\varphi(r) = -\frac{r}{\rho} + 1 + \rho + \frac{1}{\rho}$ for $1 + \rho^2 \leqslant r \leqslant 1 + \rho^2 + \rho$,
(ii) $|\varphi(r)| \leqslant C$, $|\varphi'(r)| \leqslant 1/\rho$, $|\varphi''(r)| \leqslant C(\rho)$ for all $r > 0$.

Analogously to [10], we start with a set

$$\begin{aligned} F_\delta := \bigl\{x \in \mathbb{R}^m:\ &\mu\bigl(B(x,r)\bigr)/r^s \leqslant 2\theta^{s,*}_\mu(x) \text{ for } r \leqslant r_0,\ \theta^{s,*}_\mu(x) \leqslant C_0, \\ &\text{and } \bigl|R^{s,\varphi}_\varepsilon\mu(x) - R^{s,\varphi}_{2\varepsilon}\mu(x)\bigr| \leqslant \delta \text{ for all } 0 < \varepsilon < \varepsilon_0\bigr\}, \end{aligned} \tag{19}$$

where $0 < \delta < 1$ and C_0, r_0, ε_0 are some positive constants.

One can see that both the conditions (15) and (16) imply that

$$\lim_{\varepsilon\to 0}\bigl|R^{s,\varphi}_\varepsilon\mu(x) - R^{s,\varphi}_{2\varepsilon}\mu(x)\bigr| = 0, \quad \mu\text{-a.e. } x \in \mathbb{R}^n. \tag{20}$$

Indeed,

$$R^{s,\varphi}_\varepsilon\mu(x) = \int \int\limits_{0<t<\frac{|x-y|^2}{\varepsilon^2}} \varphi'(t)\,\mathrm{d}t\,\frac{x-y}{|x-y|^{s+1}}\,\mathrm{d}\mu(y) = \int\limits_0^{1+\rho+2\rho^2} \varphi'(t) R^s_{\varepsilon\sqrt{t}}\mu(x)\,\mathrm{d}t, \tag{21}$$

so that (16) directly gives (20). Furthermore, (21) entails that

$$\bigl|R^{s,\varphi}_\varepsilon\mu(x) - R^{s,\varphi}_{2\varepsilon}\mu(x)\bigr| \leqslant C(\rho) \int\limits_0^{1+\rho+2\rho^2} \bigl|R^s_{\varepsilon\sqrt{t},2\varepsilon\sqrt{t}}\mu(x)\bigr|\,\mathrm{d}t \leqslant C(\rho)\left(\int\limits_0^{\varepsilon\sqrt{1+\rho+2\rho^2}} \bigl|R^s_{u,2u}\mu(x)\bigr|^2\frac{\mathrm{d}u}{u}\right)^{1/2},$$

where we used the change of variables $u := \varepsilon\sqrt{t}$ and Hölder's inequality. Hence, (15) also leads to (20).

Therefore, for sufficiently small ε_0 and r_0 and sufficiently large C_0 the set F_δ has $\mu(F_\delta) > 0$. Note that $\mu(B(x,r)) \leqslant Mr^s$ for all $x \in F_\delta$, $r > 0$ and $M = \max\{2C_0, \mu(\mathbb{R}^m)/r_0^s\}$.

Let $\theta^s(x,r)$ denote the average s-dimensional density of the ball $B(x,r)$, $x \in \mathbb{R}^m$, $r > 0$, that is, $\theta^s(x,r) := \mu(B(x,r))/r^s$. We start with the following estimates.

Proposition 2.1. *(See [10].) Assume that for some $C'>0$, $r>0$ and $x_0\in\mathbb{R}^m$ we have $\mu(B(x_0,r))\geqslant C'r^s$, and denote by n the biggest integer strictly smaller than s. Then for a sufficiently small ρ (depending on s only) and any $\tau\in(0,1/20)$ there exist a constant $\omega_0=C(C',M,\rho)\tau^{-s\frac{1}{\log_4(1+\rho^2/4)}}$, $\varepsilon\in(\frac{r}{\tau},\omega_0\frac{r}{\tau}]$ and a set of points $y_0,\dots,y_{n+1}\in B(x_0,r)\cap F_\delta$ such that*

$$\theta^s(y_0,4\varepsilon)\leqslant C(\rho)\theta^s(y_0,\varepsilon),\qquad \theta^s(y_0,\varepsilon)\geqslant C'\tau^s/2, \tag{22}$$

and

$$\sum_{j=1}^{n+1}\bigl|R_\varepsilon^{s,\varphi}\mu(y_j)-R_\varepsilon^{s,\varphi}\mu(y_0)\bigr|+\theta^s(y_0,3\varepsilon)\frac{r^2}{\varepsilon^2}\geqslant C\bigl(C',M,s\bigr)(n+1-s)r\frac{\theta^s(y_0,\varepsilon)}{\varepsilon}. \tag{23}$$

3. The proof of the main result

We will argue by contradiction. We initially assume that $s\notin\mathbb{Z}$, and then show that the estimate in (23) is accompanied by the corresponding bound from above in terms of δ, r, τ and ε. Ultimately, choosing r, δ, τ sufficiently small leads to a contradiction. Observe that in the case $s\in\mathbb{Z}$ the lower bound in (23) is degenerate, and hence, such an argument could not be constructed.

Set

$$\delta:=\frac{\tau^{s+2}}{\omega_0}=C(M,\rho)\tau^{s+2+s\frac{1}{\log_4(1+\rho^2/4)}}, \tag{24}$$

where the constant $C(M,\rho)$ is equal to the reciprocal of $C(C',M,\rho)$ from the definition of ω_0 corresponding to $C'=1/2$. Note that M depends on r_0 and C_0 in the definition of F_δ, however, the choice of r_0 and C_0 is determined solely by the properties of μ and can be made independent of δ.

Going further, fix ε_0 and take

$$r<\varepsilon_0\delta=\varepsilon_0\frac{\tau^{s+2}}{\omega_0}=C(M,\rho)\,\varepsilon_0\tau^{s+2+s\frac{1}{\log_4(1+\rho^2/4)}}\quad\text{such that } \mu\bigl(B(x_0,r)\cap F_\delta\bigr)\geqslant r^s/2. \tag{25}$$

Now that r and δ are fixed, we invoke Proposition 2.1, find the points $y_0,\dots,y_{n+1}$ and choose $\varepsilon\in(\frac{r}{\tau},\omega_0\frac{r}{\tau}]$ such that (22) and (23) are satisfied. However, for every $x,z\in B(x_0,r)\cap F_\delta$, $x\in\mathbb{R}^m$ we have

$$\bigl|R_\varepsilon^{s,\varphi}\mu(x)-R_\varepsilon^{s,\varphi}\mu(z)\bigr|\leqslant C(\rho)M\delta+C\delta\log\frac{r}{\delta\varepsilon}, \tag{26}$$

whenever $2\varepsilon<\frac{r}{\delta}<\varepsilon_0$. Indeed, a direct calculation shows that for $\eta\in[\frac{r}{2\delta},\frac{r}{\delta}]$

$$\bigl|R_\eta^{s,\varphi}\mu(x)-R_\eta^{s,\varphi}\mu(z)\bigr|\leqslant C(\rho)(r/\delta)^{-s-1}|z-x|\mu\bigl(B(x_0,4r/\delta)\bigr)\leqslant C(\rho)M\delta. \tag{27}$$

Then we can choose $\eta\in[\frac{r}{2\delta},\frac{r}{\delta}]$ such that $\eta=2^k\varepsilon$ for some $k\in\mathbb{N}$, so that

$$\begin{aligned}&\bigl|R_\varepsilon^{s,\varphi}\mu(x)-R_\varepsilon^{s,\varphi}\mu(z)\bigr|\\&\quad\leqslant\bigl|R_\varepsilon^{s,\varphi}\mu(x)-R_\eta^{s,\varphi}\mu(x)\bigr|+\bigl|R_\eta^{s,\varphi}\mu(x)-R_\eta^{s,\varphi}\mu(z)\bigr|+\bigl|R_\eta^{s,\varphi}\mu(z)-R_\varepsilon^{s,\varphi}\mu(z)\bigr|\\&\quad\leqslant C(\rho)M\delta+C\sup_{x\in F_\delta}\sup_{1\leqslant i\leqslant k}\bigl|R_{2^i\varepsilon}^{s,\varphi}\mu(x)-R_{2^{i-1}\varepsilon}^{s,\varphi}\mu(x)\bigr|\log\frac{r}{\delta\varepsilon}\leqslant C(\rho)M\delta+C\delta\log\frac{r}{\delta\varepsilon}.\end{aligned} \tag{28}$$

Therefore, (23) is complemented by the estimate

$$\sum_{j=1}^{n+1}\bigl|R_\varepsilon^{s,\varphi}\mu(y_j)-R_\varepsilon^{s,\varphi}\mu(y_0)\bigr|+\theta^s(y_0,3\varepsilon)\frac{r^2}{\varepsilon^2}\leqslant C(M,\rho)\left(\delta+\delta\log\frac{r}{\varepsilon\delta}+\theta^s(y_0,\varepsilon)\frac{r^2}{\varepsilon^2}\right), \tag{29}$$

where we used (26) and (22). Now combining (23) with (29) and dividing both sides by r/ε we arrive at the estimate

$$\theta^s(y_0,\varepsilon)\leqslant C(M,s,\rho)\left(\frac{\delta\varepsilon}{r}+\frac{\delta\varepsilon}{r}\log\frac{r}{\varepsilon\delta}+\theta^s(y_0,\varepsilon)\frac{r}{\varepsilon}\right). \tag{30}$$

According to our choice of δ and r,

$$\frac{\delta\varepsilon}{r} \leqslant \frac{\tau^{s+2}}{\omega_0}\frac{\omega_0}{\tau} = \tau^{s+1} \leqslant C\tau\,\theta^s(y_0,\varepsilon) \quad \text{and} \quad \frac{r}{\varepsilon} \leqslant \tau. \tag{31}$$

Now (30) and (31) give the bound

$$\theta^s(y_0,\varepsilon) \leqslant C(M,s,\rho)\big(\tau+\tau^{1-\alpha}\big)\theta^s(y_0,\varepsilon), \quad \forall\alpha>0, \tag{32}$$

which for $\tau > 0$ sufficiently small leads to a contradiction. □

References

[1] G. David, S. Semmes, Singular integrals and rectifiable sets in R^n: Beyond Lipschitz graphs, Astérisque 193 (1991) 152.
[2] G. David, S. Semmes, Analysis of and on Uniformly Rectifiable Sets, Mathematical Surveys and Monographs, vol. 38, American Mathematical Society, Providence, RI, 1993.
[3] P. Mattila, M. Melnikov, J. Verdera, The Cauchy integral, analytic capacity, and uniform rectifiability, Ann. of Math. (2) 144 (1) (1996) 127–136.
[4] F. Nazarov, S. Treil, A. Volberg, Weak type estimates and Cotlar inequalities for Calderón–Zygmund operators on nonhomogeneous spaces, Internat. Math. Res. Notices 9 (1998) 463–487.
[5] F. Nazarov, S. Treil, A. Volberg, The *Tb*-theorem on non-homogeneous spaces, Acta Math. 190 (2) (2003) 151–239.
[6] L. Prat, Potential theory of signed Riesz kernels: Capacity and Hausdorff measure, Internat. Math. Res. Notices 19 (2004) 937–981.
[7] L. Prat, Principal values for the signed Riesz kernels of non-integer dimensions, preprint, 2006, as cited in [10].
[8] X. Tolsa, Principal values for the Cauchy integral and rectifiability, Proc. Amer. Math. Soc. 128 (7) (2000) 2111–2119.
[9] X. Tolsa, Xavier growth estimates for Cauchy integrals of measures and rectifiability, Geom. Funct. Anal. 17 (2) (2007) 605–643.
[10] X. Tolsa, A. Ruiz de Villa, Non existence of principal values of signed Riesz transforms of non integer dimension, preprint, 2008.
[11] M. Vihtilä, The boundedness of Riesz s-transforms of measures in $\mathbb{R}^n$, Proc. Amer. Math. Soc. 124 (12) (1996) 3797–3804.
[12] A. Volberg, Calderón–Zygmund Capacities and Operators on Nonhomogeneous Spaces, CBMS Regional Conference Series in Mathematics, vol. 100, American Mathematical Society, Providence, RI, 2003, published for the Conference Board of the Mathematical Sciences, Washington, DC.