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Dynamical Systems/Mathematical Problems in Mechanics

Stability of relative equilibria and Morse index of central configurations

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Abstract

For the planar n-body problem, if the Morse index or the nullity of a central configuration as a critical point of Newton potential function restricted on the "shape sphere" is odd, then the relative equilibrium corresponding to the central configuration is linearly unstable. ***To cite this article: X. Hu, S. Sun, C. R. Acad. Sci. Paris, Ser. I 347 (2009).***

Résumé

Stabilité d'équilibre relatif et indice de Morse de configuration centrale. Dans le problème plan des n corps, si l'indice de Morse ou la nullité d'une configuration centrale vue comme un point critique du potentiel newtonien restreint à la « sphère des formes » est impair, l'équilibre relatif correspondant est linéairement instable. ***Pour citer cet article : X. Hu, S. Sun, C. R. Acad. Sci. Paris, Ser. I 347 (2009).***

1. Introduction

Newton's equation of motion of the planar n-body problem is

$$M\ddot{q} = \nabla U(q), \tag{1}$$

where $M = \mathrm{diag}(m_1, m_1, \ldots, m_n, m_n)_{2n\times 2n}$ with $m_i \in \mathbf{R}^+$ the mass of the i-th celestial body, $q = (q_1, \ldots, q_n)^T$ with $q_i \in \mathbf{R}^2$ its position, and $U(q) = \sum_{1\leqslant i<j\leqslant n} \frac{m_i m_j}{\|q_i - q_j\|}$ is the potential function. It is a system of second-order ordinary differential equations defined on the configuration space $X = (\mathbf{R}^2)^n \setminus \Delta$ with Δ collision set $\Delta = \{q \in (\mathbf{R}^2)^n \mid q_i = q_j,$ for some $i \neq j\}$.

It is well known that the corresponding Hamiltonian equation is the following first-order system of ordinary differential equations in phase space $T^*X \cong X \times (R^2)^n$: $\dot{q} = \frac{\partial H}{\partial p} = M^{-1}p^T$, $\dot{p} = -\frac{\partial H}{\partial q} = \nabla U(q)$ with the energy Hamiltonian $H = \frac{1}{2}pM^{-1}p^T - U(q)$.

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Central configuration is an important and natural concept in the n-body problem. It is defined by the following algebraic equations with real coefficients:

$$\lambda M q = -\nabla U(q), \tag{2}$$

where $\lambda = U(q)/I(q)$ is some positive constant with $I(q) = q^T M q$ the moment of inertia. For further properties on central configurations, please refer to [1] and references therein. Following Smale ([9], see also [5]), we have a Morse-theoretic interpretation for the central configurations. Let $S = \{q \in X = (\mathbf{R}^2)^n \setminus \Delta \mid I(q) = \sum_{i=1}^n m_i \|q_i\|^2 = 1, \sum_{i=1}^n m_i q_i = 0\}$. It is a noncompact $(2n-3)$-dimensional manifold. The potential function $U(q)$ restricted to S is S^1-invariant, and its critical points are central configurations. Its Hessian at central configuration q is $H_{\mathrm{ess}}(q) = (D^2U(q) + U(q)M)|_{T_q(S)}$. We denote by $\mathcal{M}(q)$ the Morse index of q, i.e., the dimension of negative definite subspace of $H_{\mathrm{ess}}(q)$ and denote its nullity by $\mathcal{V}(q) + 1 = \dim\ker(H_{\mathrm{ess}}(q))$. Let $\hat{S} = S/S^1$, which we call the "shape sphere". It is a noncompact $2(n-2)$-dimensional manifold. Usually the critical point q of the reduced potential function $U(q)|_{\hat{S}}$ is still called central configuration with Morse index $\mathcal{M}(q)$ and nullity $\mathcal{V}(q)$. There is a vast literature on the estimates of numbers of central configurations and their Morse indices ([3,5,7] and references therein).

Given a central configuration q, we can construct a relative equilibrium, the circular periodic solution of the planar n-body problem. Let $A(t)$ be the $(2n \times 2n)$-matrix with n diagonal blocks of the form

$$\begin{pmatrix} \cos(kt) & -\sin(kt) \\ \sin(kt) & \cos(kt) \end{pmatrix}, \quad k^2 = \lambda = \frac{U(q)}{I(q)}, \tag{3}$$

then the solution is of the form $A(t)q$. We are interested in the linear stability of this kind of rigid motion related to a central configuration of the planar n-body problem.

As pointed out in [5], there are two four-dimensional invariant symplectic subspaces E_1 and E_2, and they are associated to the translation symmetry, dilation and rotation symmetry of the system. In other words [4], there is a symplectic coordinate system in which the linearized system of the planar n-body problem decouples into three subsystems on E_1, E_2 and $E_3 = (E_1 \cup E_2)^\perp$, where $\perp$ denotes the symplectic orthogonal complement. By linear stability we mean the monodromy matrix M restricted to E_3 is linearly stable, that is $M|_{E_3}$ is semi-simple and its spectrum is in the unit circle $\mathbf{U}$ on the complex plane $\mathbf{C}$. Our main theorem is the following:

Theorem. *If the Morse index or the nullity of a central configuration as a critical point of the potential function $U|_{\hat{S}}$ is odd, then the corresponding relative equilibrium is linearly unstable.*

Eq. (2) of central configuration in $\hat{S}$ can be written $F(q) = M^{-1}\nabla U(q) + U(q)q = 0$. This is the gradient vector field of the restriction $U|_{\hat{S}}$ with respect to the mass metric $\langle q, q\rangle = q^T M q$.

Moeckel conjectures, unpublished, that only central configurations which are minima could possibly give rise to linearly stable relative equilibria. It is true for three bodies and planar $(1+n)$-body problem [6]. This conjecture raises the question on the relationship between the two dynamics: gradient flow on the shape sphere and Newton's equations on phase space. This is our motivation, and the theorem is our first attempt to understand the relationship between these two dynamics.

For more works on the stability of relative equilibria, please refer to [5,8] and references therein.

2. Proof of the theorem

Let $J = \left(\begin{smallmatrix} 0 & -I_n \\ I_n & 0 \end{smallmatrix}\right)$ with I_n the standard identity matrix on $\mathbf{R}^n$. We denote by $\mathrm{Sp}(2n) = \{M \in GL(2n) \mid M^T J M = J\}$ the symplectic group on $(\mathbf{R}^{2n}, \omega = \sum \mathrm{d}x_i \wedge \mathrm{d}y_i)$.

We recall some facts about the Krein form [2, pp. 10–13]. Let $G = \sqrt{-1}J$, then the *Krein form* is defined to be $\langle Gx, y\rangle$, $\forall x, y \in \mathbf{C}^{2n}$, with standard Hermitian inner product $\langle\cdot,\cdot\rangle$ in $\mathbf{C}^{2n}$. Let $V_\lambda(M) = \ker(M - \lambda I)^{2n}$, $M \in \mathrm{Sp}(2n)$. For $\lambda, \mu \in \sigma(M)$, the spectrum of M, if $\lambda\bar{\mu} \neq 1$, then V_λ and V_μ are G-orthogonal. This shows that for $\lambda \in \sigma(M) \cap \mathbf{U}$ with $\mathbf{U}$ the unit circle on the complex plane, $G|_{V_\lambda}$ is non-degenerate. In this case, denote the total multiplicities of positive and negative eigenvalues of $G|_{V_\lambda}$ by (p_λ, q_λ) respectively. The pair (p_λ, q_λ) of integers is called *Krein-type number* of $\lambda \in \sigma(M) \cap \mathbf{U}$. It has the property that

$$(p_{\bar{\lambda}}, q_{\bar{\lambda}}) = (q_\lambda, p_\lambda), \quad p_1 = q_1, \quad p_{-1} = q_{-1}. \tag{4}$$

Suppose B is a symmetric matrix on $\mathbf{R}^{2n}$, $t \in \mathbf{R}$, then $M = \exp(JBt) \in \mathrm{Sp}(2n)$. From the properties of symplectic matrix, we know $\lambda \in \sigma(M)$ implies that $\bar{\lambda}$, λ^{-1} and $\bar{\lambda}^{-1}$ are all eigenvalues of M and have the same geometric and algebraic multiplicities as λ. It is obvious that for t small enough $\lambda \in \sigma(JB)$ if and only if $e^{\lambda t} \in \sigma(M)$, and $\ker(JB - \lambda I_{2n}) = \ker(M - e^{\lambda t} I_{2n})$, $V_\lambda(JB) = V_{e^{\lambda t}}(M)$. This shows that, if $\lambda \in \sigma(JB)$, then $\bar{\lambda}$, $-\lambda$ and $-\bar{\lambda}$ are eigenvalues of JB with the same geometric and algebraic multiplicities.

JB is called linearly stable if it is semi-simple and all its eigenvalues are on the imaginary axis. Obviously, $M = \exp(JBt)$ is linearly stable if and only if JB is linearly stable. Suppose M is linearly stable. Semi-simplicity means that, for each eigenvalue, its algebraic multiplicity and geometric multiplicity are the same. Moreover, for $\lambda \in \sigma(JB)$ on the imaginary axis, $G|_{V_\lambda}$ is non-degenerate and $\sum_\lambda \dim V_\lambda = 2n$. By (4), the multiplicity of the possible eigenvalue 0 must be even, thereby the signature of $G|_{V_0}$ is zero.

It is quite convenient to work in rotating frame since we are only interested in circular orbits in this paper. Here we follow Moeckel [5, p. 85].

Let k be the rotational velocity of the circular solution as in (3), and introduce new coordinates $Q = A(t)q$, $P^T = A(t)p^T$. Now the Hamiltonian system becomes:

$$\dot{Q} = M^{-1}P^T + K_n Q, \qquad \dot{P}^T = \nabla U(Q) + K_n P^T, \tag{5}$$

where K_n is $(2n \times 2n)$-matrix with n diagonal blocks of the form $\left(\begin{smallmatrix} 0 & -k \\ k & 0 \end{smallmatrix}\right)$. Note that K_n is anti-symmetric. It is a Hamiltonian system with Hamiltonian $H(Q,P) = \frac{1}{2}PM^{-1}P^T + PK_nQ - U(Q)$. Now the circular periodic solution is a rest point, and we still denote it by Q by a little abusing notations. The linearized Hamiltonian system at the rest point Q is $\dot{z} = -JB(Q)z$, where the symmetric matrix $B(Q)$ is $\left(\begin{smallmatrix} -D^2U(Q) & K_n^T \\ K_n & M^{-1} \end{smallmatrix}\right)$.

As in [4, p. 266, Corollary 2.1] and [5, pp. 92–93], (5) can be decomposed into 3 subsystems on E_1, E_2 and $E_3 = (E_1 \cup E_2)^\perp$ respectively. The basis of E_1 is $(u,0)$, $(0,Mu)$, $(v,0)$, $(0,Mv)$, where $u = (1,0,1,0,\ldots)$, $v = (0,1,0,1,\ldots)$, and E_2 is spanned by $(Q,0)$, $(0,MQ)$, $(K_nQ,0)$, $(0,K_nMQ)$. For $X = (g,z,w) \in \mathbf{R}^2 \times \mathbf{R}^2 \times \mathbf{R}^{2n-4}$ and $Y = (G,Z,W) \in \mathbf{R}^2 \times \mathbf{R}^2 \times \mathbf{R}^{2n-4}$, we do the linear symplectic transformation of the form $Q = CX$, $P = C^{-T}Y$, where C is such that $C^{-1}K_nC = K_n$, $C^TMC = I$ [4, p. 263]. Now $B(Q)$ in this new coordinate system has the form $B(Q) = B_1 \oplus B_2 \oplus B_3$, where $B_i = B|_{E_i}$. The essential part B_3 of the stability problem is a $(4n-8) \times (4n-8)$-matrix of the form $B_3 = \left(\begin{smallmatrix} -\mathcal{D} & K_{n-2}^T \\ K_{n-2} & I_{2(n-2)} \end{smallmatrix}\right)$, where $\mathcal{D} = C^T D^2U(Q)C|_{w \in \mathbf{R}^{2n-4}}$. It is easy to see that

$$\begin{pmatrix} I_{2(n-2)} & K_{n-2} \\ 0 & I_{2(n-2)} \end{pmatrix} \begin{pmatrix} -\mathcal{D} & K_{n-2}^T \\ K_{n-2} & I_{2(n-2)} \end{pmatrix} \begin{pmatrix} I_{2(n-2)} & 0 \\ -K_{n-2} & I_{2(n-2)} \end{pmatrix} = \begin{pmatrix} -(\mathcal{D} + U(Q)) & 0 \\ 0 & I_{2(n-2)} \end{pmatrix}. \tag{6}$$

Note that $\mathcal{D} + U(Q) = C^T D^2U(Q)C|_{w \in \mathbf{R}^{2n-4}} + U(Q) = C^T(D^2U(Q) + U(Q)M)C|_{w \in \mathbf{R}^{2n-4}}$, it is exactly the Hessian of $U|_{\hat{S}}$ with nullity $\mathcal{V}(Q)$ and Morse index $\mathcal{M}(Q)$.

We prove the theorem by contradiction. Since $\lambda \in \sigma(JB_3)$ if and only if $\ker(B_3 + \lambda J) \neq 0$. Suppose JB_3 is linearly stable. We consider a path of self-adjoint matrices $D_s = B_3 + sG$, $s \in [0, +\infty)$.

Since $\ker(D_0) \cong \ker(JB_3)$, the linear stability of JB_3 implies that $\dim\ker(D_0)$ is even-dimensional, the Krein operator G restricted to $\ker(D_0)$ is non-degenerate and its signature on $\ker(D_0)$ is zero. The Morse index of D_s for $0 < s \leqslant \varepsilon$, ε some small enough fixed number, is $2n - 4 - (\mathcal{M}(Q) + \mathcal{V}(Q)/2)$. On the other hand, for s large enough D_s is non-degenerate and its signature is the same as that of G, which is equal to zero. Let $\mathcal{X}$ be the total number of multiplicities of eigenvalues of GB_3 on the interval $[\varepsilon, +\infty)$. By the linear stability assumption, we have

$$\mathcal{X} + \mathcal{V}(Q)/2 = 2n - 4. \tag{7}$$

At each crossing $\lambda_0 \in [\varepsilon, +\infty)$, that is $\ker(D_{\lambda_0}) \neq 0$, G restricted to $\ker(D_{\lambda_0})$ is non-degenerate, and

$$\mathrm{sign}(G|_{\ker(D_{\lambda_0})}) = \dim\ker(D_{\lambda_0}), \quad \bmod 2 \tag{8}$$

with one side the difference of dimensions of the positive definite subspace and negative definite subspace of G on $\ker(D_{\lambda_0})$, the other side the sum. Take the sum over all crossings $\lambda_0 \in [\varepsilon, +\infty)$, we have

$$\begin{aligned} \mathcal{M}(Q) + \mathcal{V}(Q)/2 &= -\sum_{\lambda_0} \mathrm{sign}(G|_{\ker(D_{\lambda_0})}) = \sum_{\lambda_0} \dim\ker(D_{\lambda_0}) \quad \big(\bmod 2,\ \text{by (8)}\big) \\ &= \mathcal{X} \pmod 2 = 2n - 4 - \mathcal{V}(Q)/2 \quad \big(\bmod 2,\ \text{by (7)}\big) = \mathcal{V}(Q)/2 \pmod 2. \end{aligned}$$

This shows that $\mathcal{M}(Q)$ must be even, hence a contradiction. The proof is complete. $\square$

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