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Testing for equality of means of a Hilbert space valued random variable

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Abstract

We propose a test for equality of means of a random variable valued into a real separable Hilbert space. The test statistic is based on projections of empirical means onto spaces spanned by principal directions obtained from principal component analysis of the random variable. The asymptotic distribution of this test statistic is derived under the null hypothesis and the consistency of the obtained test is proved. An application to the case of functional variables is indicated. ***To cite this article: J.G. Aghoukeng Jiofack, G.M. Nkiet, C. R. Acad. Sci. Paris, Ser. I 347 (2009).***

Résumé

Un test d'égalité des moyennes d'une variable aléatoire à valeurs dans un espace de Hilbert. Nous proposons un test d'égalité des moyennes pour une variable aléatoire à valeurs dans un espace de Hilbert réel séparable. La statistique de test est basée sur des projections des moyennes empiriques sur des directions obtenues à partir de l'analyse en composantes principale de la variable aléatoire. La loi limite, sous hypothèse nulle, de cette statistique de test est obtenue et la convergence du test obtenu est prouvée. L'application au cas de variables fonctionnelles est indiquée. ***Pour citer cet article : J.G. Aghoukeng Jiofack, G.M. Nkiet, C. R. Acad. Sci. Paris, Ser. I 347 (2009).***

Version française abrégée

Soient X et Y deux variables aléatoires (v.a.) à valeurs respectivement dans $(H, \mathcal{B}_H)$ et $(F, \mathcal{P}(F))$, où H est un espace de Hilbert réel séparable, $\mathcal{B}_H$ est sa tribu borélienne, F est l'ensemble $\{1, \dots, q\}$ et $\mathcal{P}(F)$ la tribu de ses parties. Notant $\|\cdot\|$ la norme induite par le produit scalaire, supposant que $\mathbb{E}(\|X\|^4) < +\infty$ et considérant $m = \mathbb{E}(X)$, $m_\ell = \mathbb{E}(X|Y = \ell)$, nous proposons dans cette note un test de l'hypothèse nulle $\mathcal{H}_0$ définie en (1) contre l'hypothèse alternative notée $\mathcal{H}_1$. Pour cela, on suppose que les valeurs propres décroissantes λ_i de l'opérateur de covariance V de X sont distinctes, et on utilise la statistique définie en (2) sur la base d'un échantillon i.i.d. $\{(X_i, Y_i)\}_{1 \leqslant i \leqslant n}$

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de (X, Y). Considérant les matrices diagonales $D_V(p) = \mathrm{diag}(\lambda_i; 1 \leqslant i \leqslant p)$ et $\Delta = \mathrm{diag}(p_\ell; 1 \leqslant \ell \leqslant q)$, où $p_\ell = P(Y = \ell) > 0$, la matrice carrée Σ d'ordre pq définie en (3), et notant $\otimes^K$ le produit matriciel de Kronecker, on a :

Théorème 2.1. *Sous $\mathcal{H}_0$, $n\widehat{T}_n(p) \xrightarrow{\mathcal{L}} \mathcal{Q} = U^T \Gamma U$ $(n \to +\infty)$, où $\Gamma = (D_V(p))^{-1} \otimes^K \Delta^{-1}$ et $U \rightsquigarrow N_{\mathbb{R}^{pq}}(0, \Sigma)$.*

Pour un niveau $\alpha \in]0, 1[$ donné, $\mathcal{H}_0$ est rejetté lorsque $\mathbb{F}_\mathcal{Q}(n\widehat{T}_n(p)) > 1 - \alpha$, où $\mathbb{F}_\mathcal{Q}$ désigne la fonction de répartition de $\mathcal{Q}$. Celle-ci peut être calculée en utilisant des formules d'approximation données dans [13] et qui nécessitent le calcul des valeurs propres de $\Sigma^{1/2}\Gamma\Sigma^{1/2}$. Les matrices Σ and Γ étant inconnues en pratique, on les remplace par les estimateurs convergents obtenus en remplaçant, dans les formules qui les définissent, chaque paramètre par son analogue empirique. Le théorème suivant montre la convergence du test proposé, pour des valeurs de p suffisamment grandes.

Théorème 2.2. *Sous $\mathcal{H}_1$, il existe $p_0 \in \mathbb{N}^*$ tel que pour tous $p \geqslant p_0$ et $\alpha \in]0, 1[$, on a :*

$$\lim_{n\to+\infty} P\big(\mathbb{F}_\mathcal{Q}\big(n\widehat{T}_n(p)\big) > 1 - \alpha\big) = 1.$$

Ce test peut être appliqué au cas d'une variable fonctionnelle. En effet, lorsque $X = (X(t))_{t\in[0,1]}$ est à valeurs dans $L^2([0, 1])$, et si l'on suppose que chaque X_i est observé en des points $t_1, \ldots, t_L$ tels que $0 \leqslant t_1 < t_2 < \cdots < t_L \leqslant 1$, on obtient la matrice $\mathbb{X} = (X^{(i)}(t_j))$ sur la base de laquelle on peut calculer les $\hat{\lambda}_i$ et les composantes principales $\hat{c}_i(k) = \langle X_i, \hat{u}_k\rangle$ en utilisant, par exemple, la fonction pca.fd disponible dans la librairie fda de R. Cela permet d'obtenir les matrices $\hat{\Gamma}$ et $\hat{\Sigma}$; cette dernière étant construite comme en (3) avec les termes $\hat{\sigma}_{ijk\ell}$ donnés en (6).

1. Introduction

Letting $(\Omega, \mathcal{A}, P)$ be a probability space, we consider two random variables X and Y defined on this space and valued respectively into $(H, \mathcal{B}_H)$ and $(F, \mathcal{P}(F))$, where H is a real separable Hilbert space, $\mathcal{B}_H$ is the related Borel σ-field, F is the set $\{1, \ldots, q\}$ and $\mathcal{P}(F)$ denotes the σ-field of all subsets of F. Denoting by $\langle\cdot,\cdot\rangle$ the inner product of H and by $\|\cdot\|$ the related norm, we suppose that $\mathbb{E}(\|X\|^4) < +\infty$. Then we consider the mean $m = \mathbb{E}(X)$ and, for any $\ell \in F$, the conditional mean $m_\ell = \mathbb{E}(X|Y = \ell)$ and the probability $p_\ell = P(Y = \ell)$ which is assumed, without loss of generality, to be non-null. We are interested in testing for the hypothesis

$$\mathcal{H}_0 : m_1 = \cdots = m_q \tag{1}$$

against the alternative $\mathcal{H}_1$: $\exists \ell \in F$, $\tau_\ell \neq 0$, where $\tau_\ell = m_\ell - m$. This problem is within a field, called *Functional Data Analysis*, that has emerged in the last decade. It consists of statistical methods allowing analysis of data that are curves, including finely sampled records over some time interval. The book by Ramsay and Silverman [14] gives a comprehensive introduction to this field of statistics, whereas nonparametric approaches are tackled in [9]. Several recent papers deal with statistical testing procedures for functional variables, mainly in the context of functional linear regression models for which tests for no effect are considered (see, e.g., [3,4,11]). However, a few papers propose tests on the means of functional variables. More precisely, Mas [12] introduced a procedure for testing the equality of the mean to a fixed value, whereas tests for comparison of means, that is the hypothesis $\mathcal{H}_0$ given in (1), were proposed in [2,5] and [8]. The practical interest of using these methods was illustrated in [2] on electroencephalogram data, and in [5] on data from experimental cardiology. Recent works also show the interest of making statistical inference on others location parameters in the context of functional variables. In this vein, mode and median were tackled in [9], whereas a general depth measure is proposed in [6] which also contains a discussion about the applicability of this idea for inference in functional data analysis. Our purpose in this paper is to introduce a new method for testing for $\mathcal{H}_0$; this method is based on projections of the τ_ℓ's onto principal directions, that is lines spanned by unit eigenvectors of the covariance operator of X. Our approach is described in the general framework of Hilbert space valued random variables and is easily applied when functional variables are considered.

2. Main results

The covariance operator of X is defined by $V = \mathbb{E}((X - m) \otimes (X - m))$, where $\otimes$ denotes the tensor product such that, for any (x, y), $x \otimes y$ is the linear map: $h \mapsto \langle x, h\rangle y$. It is known that, under the above assumption, V is a Hilbert–

Schmidt self-adjoint operator acting from H to itself. Letting $(\lambda_i)_{i\geqslant 1}$ be the decreasing sequence of eigenvalues of V, we consider an orthonormal basis $(u_i)_{i\geqslant 1}$ of H such that u_i is an eigenvalue of V associated with λ_i. We assume throughout the paper that the following assumption holds:

$$(\mathcal{A})\quad \forall i \geqslant 1,\ \lambda_i > \lambda_{i+1} > 0.$$

Given an i.i.d. sample $\{(X_i, Y_i)\}_{1\leqslant i\leqslant n}$ of (X, Y), we consider the empirical counterpart $\hat{V}_n$ of V, defined as $\hat{V}_n = n^{-1}\sum_{i=1}^{n}(X_i - \bar{X}^{(n)}) \otimes (X_i - \bar{X}^{(n)})$, where $\bar{X}^{(n)} = n^{-1}\sum_{i=1}^{n} X_i$, and we denote by $(\hat{\lambda}_i, \hat{u}_i)_{i\geqslant 1}$ a family of eigenelements of $\hat{V}_n$ such that $(\hat{\lambda}_i)_{i\geqslant 1}$ is the decreasing sequence of eigenvalues of $\hat{V}_n$ and $(\hat{u}_i)_{i\geqslant 1}$ is an orthonormal system of related eigenvectors, $\hat{u}_i$ being associated with $\hat{\lambda}_i$. In addition, for $\ell \in F$, we put

$$\hat{n}_\ell = \sum_{i=1}^{n} \mathbf{1}_{\{Y_i=\ell\}}, \quad \hat{p}_\ell = \frac{\hat{n}_\ell}{n}, \quad \bar{X}_\ell^{(n)} = \frac{1}{\hat{n}_\ell}\sum_{i=1}^{n} \mathbf{1}_{\{Y_i=\ell\}} X_i \quad \text{and} \quad \hat{\tau}_\ell^n = \bar{X}_\ell^{(n)} - \bar{X}^{(n)}.$$

For testing $\mathcal{H}_0$ against $\mathcal{H}_1$, we make use of the statistic given by

$$\widehat{T}_n(p) = \sum_{i=1}^{p}\sum_{\ell=1}^{q} \hat{\lambda}_i^{-1}\hat{p}_\ell\langle\hat{\tau}_\ell^n, \hat{u}_i\rangle^2, \tag{2}$$

where p is sufficiently large. This statistic is a consistent estimator of $T(p) = \sum_{i=1}^{p}\sum_{\ell=1}^{q}\lambda_i^{-1}p_\ell\langle\tau_\ell, u_i\rangle^2$. Indeed, an obvious application of the strong law of large numbers gives the almost sure convergence of $\hat{p}_\ell$ (respectively $\hat{\tau}_\ell^n$; respectively $\hat{V}_n$) to p_ℓ (respectively τ_ℓ; respectively V in $\mathcal{L}(H)$) as $n \to +\infty$. Further, Lemma 1 in [10] gives $|\hat{\lambda}_i - \lambda_i| \leqslant \|\hat{V}_n - V\|$ and $\|\hat{u}_i - u_i\| \leqslant 2\sqrt{2}\|\hat{u}_i \otimes \hat{u}_i - u_i \otimes u_i\|$. Then, using Proposition 3 in [7] which ensures the almost sure convergence in $\mathcal{L}(H)$ of $\hat{u}_i \otimes \hat{u}_i$ to $u_i \otimes u_i$ as $n \to +\infty$, we deduce from the preceding inequalities that $\hat{\lambda}_i$ and $\hat{u}_i$ converge almost surely, as $n \to +\infty$, to λ_i and u_i respectively. Consequently, $\widehat{T}_n(p)$ converges almost surely to $T(p)$ as $n \to +\infty$. The asymptotic distribution under $\mathcal{H}_0$ of this statistic is given in the theorem that is stated below. Let us introduce the operators

$$V^{(\ell)} = \mathbb{E}\big((X - m_\ell) \otimes (X - m_\ell)\big|Y = \ell\big) \quad \text{and} \quad \Theta_{j\ell} = \delta_{j\ell}p_jV^{(j)} + p_jp_\ell\big(V - V^{(j)} - V^{(\ell)}\big),$$

where $(j, \ell) \in F^2$ and $\delta_{j\ell}$ denotes the Kronecker delta. This permits to consider the $pq \times pq$ matrix

$$\Sigma = \begin{pmatrix} \sigma_{1111} & \sigma_{1112} & \cdots & \sigma_{11pq} \\ \sigma_{1211} & \sigma_{1212} & \cdots & \sigma_{12pq} \\ \vdots & \vdots & \cdots & \vdots \\ \sigma_{pq11} & \sigma_{pq12} & \cdots & \sigma_{pqpq} \end{pmatrix}, \quad \text{where } \sigma_{ijk\ell} = \langle u_i, \Theta_{j\ell}u_k\rangle. \tag{3}$$

Considering the diagonal matrices $D_V(p) = \mathrm{diag}(\lambda_i; 1 \leqslant i \leqslant p)$ and $\Delta = \mathrm{diag}(p_\ell; 1 \leqslant \ell \leqslant q)$, and denoting by $\otimes^K$ the Kronecker matrix product, we have:

Theorem 2.1. *Under $\mathcal{H}_0$, $n\widehat{T}_n(p)$ converges in distribution, as $n \to +\infty$, to $\mathcal{Q} = U^T\Gamma U$, where $\Gamma = (D_V(p))^{-1} \otimes^K \Delta^{-1}$ and $U \rightsquigarrow N_{\mathbb{R}^{pq}}(0, \Sigma)$.*

Sketch of the proof. We first express the test statistic as $\widehat{T}_n(p) = \mathrm{tr}(\widehat{R}_p^n)$, where $\widehat{R}_p^n = (\widehat{V}_p^n)^{-1}\widehat{B}_n$ with $\widehat{V}_p^n = \sum_{i=1}^{p}\hat{\lambda}_i\hat{u}_i \otimes \hat{u}_i$ and $\widehat{B}_n = \sum_{\ell=1}^{q}\hat{p}_\ell^n\ \hat{\tau}_\ell^n \otimes \hat{\tau}_\ell^n$. Then, letting $\{f_1, \ldots, f_q\}$ be the canonical basis of $\mathbb{R}^q$, putting $W = \sum_{\ell=1}^{q}\mathbf{1}_{\{Y=\ell\}}f_\ell$, $W_i = \sum_{\ell=1}^{q}\mathbf{1}_{\{Y_i=\ell\}}f_\ell$ and

$$Z = \begin{pmatrix} X - m \\ W \end{pmatrix}, \qquad Z_i = \begin{pmatrix} X_i - \bar{X}^{(n)} \\ W_i \end{pmatrix}, \qquad V_Z = \mathbb{E}(Z \otimes Z), \qquad V_Z^{(n)} = \frac{1}{n}\sum_{i=1}^{n} Z_i \otimes Z_i,$$

we show that $n\widehat{R}_p^n = \varphi_n(K_n)$, where $K_n = \sqrt{n}(V_Z^{(n)} - V_Z)$ and φ_n is the random map from $\mathcal{L}(H \times \mathbb{R}^q)$ to $\mathcal{L}(H)$ given by $\varphi_n(S) = \sum_{\ell=1}^{q}(\hat{p}_\ell^n)^{-1}(\Pi_{12}(S)f_\ell) \otimes ((\widehat{V}_p^n)^{-1}(\Pi_{12}(S)f_\ell))$. Here, Π_{12} is defined by $\Pi_{12}(S) = P_1SP_2^*$ where P_1 (respectively P_2) is the canonical projection from $H \times \mathbb{R}^q$ to H (respectively $\mathbb{R}^q$). On the one hand, we show that K_n

converges in distribution as $n \to +\infty$ to a r.v. K having a centered normal distribution in $\mathcal{L}(H \times \mathbb{R}^q)$ with covariance operator equal to that of

$$\mathcal{S}\left(\frac{1}{2}\big(Z_0 \otimes Z_0 - \mathbb{E}(Z_0 \otimes Z_0)\big) - (Z_0 - \mu) \otimes \Pi_0(\mu) - \Pi_0(Z_0 - \mu) \otimes \mu + \Pi_0(Z_0 - \mu) \otimes \Pi_0(\mu)\right),$$

where $\mathcal{S}: A \in \mathcal{L}(H \times \mathbb{R}^q) \mapsto A + A^*$ and

$$Z_0 = \begin{pmatrix} X \\ W \end{pmatrix}, \qquad \mu = \mathbb{E}(Z_0), \qquad \Pi_0: \begin{pmatrix} u \\ v \end{pmatrix} \in H \times \mathbb{R}^q \longmapsto \begin{pmatrix} u \\ 0 \end{pmatrix} \in H \times \mathbb{R}^q.$$

On the other hand, using this convergence result and the almost sure convergences of $\hat{p}_\ell^n$ and $(\widehat{V}_p^n)^{-1}$ to p_ℓ and $(V_p)^{-1}$ respectively, where $V_p = \sum_{i=1}^{p} \lambda_i u_i \otimes u_i$, it is easy to check that $\varphi_n(K_n) - \varphi(K_n)$ converges in probability to 0, φ being defined by $\varphi(S) = \sum_{\ell=1}^{q} (p_\ell)^{-1} (\Pi_{12}(S) f_\ell) \otimes ((V_p)^{-1}(\Pi_{12}(S) f_\ell))$. Therefore, $\varphi_n(K_n)$ has the same asymptotic distribution than $\varphi(K_n)$. From the continuity of φ we deduce (see [1]) that this asymptotic distribution is the distribution of $\varphi(K)$, that is a centered normal distribution. Consequently, $n\widehat{T}_n(p)$ converges in distribution, as $n \to +\infty$, to $\mathcal{Q} = \mathrm{tr}(\varphi(K))$. From some easy calculations, we finally obtain $\mathcal{Q} = U^T \Gamma U$ where, denoting by vec the usual vectorization operator which stacks the columns of a matrix into one vector, and considering the $p \times q$ matrix $\mathbb{M}(K) = (\langle u_i, \Pi_{12}(K) f_j \rangle)_{1 \leqslant i \leqslant p,\, 1 \leqslant j \leqslant q}$, U is defined by $U = \mathrm{vec}(\mathbb{M}(K)^T)$. This random vector is a linear function of K and has, therefore, a centered normal distribution in $\mathbb{R}^{pq}$. Its covariance matrix is obtained from further calculations that show that it is equal to Σ. □

For a given significance level $\alpha \in]0, 1[$, the hypothesis $\mathcal{H}_0$ will be rejected if $\mathbb{F}_{\mathcal{Q}}(n\widehat{T}_n(p)) > 1 - \alpha$, where $\mathbb{F}_{\mathcal{Q}}$ denotes the cumulative distribution function of $\mathcal{Q}$. Since $\mathcal{Q}$ is a quadratic form of a normally distributed random vector, $\mathbb{F}_{\mathcal{Q}}$ can be computed or approximated by using formulas given in [13] and which involve the eigenvalues of $\Sigma^{1/2} \Gamma \Sigma^{1/2}$. In practice Σ and Γ are unknown; so, they are to be replaced by consistent estimators. For estimating Γ, we have to consider $\hat{\Gamma} = (\hat{D}_V(p))^{-1} \otimes^K \hat{\Delta}^{-1}$ where $\hat{D}_V(p) = \mathrm{diag}(\hat{\lambda}_i; 1 \leqslant i \leqslant p)$ and $\hat{\Delta} = \mathrm{diag}(\hat{p}_\ell; 1 \leqslant \ell \leqslant q)$. A consistent estimator of Σ is obtained by taking the $pq \times pq$ matrix $\hat{\Sigma}$ constructed as in (3) but with elements given by $\hat{\sigma}_{ijk\ell} = \langle \hat{u}_i, \hat{\Theta}_{j\ell} \hat{u}_k \rangle$, where

$$\hat{V}_n^{(\ell)} = \frac{1}{\hat{n}_\ell} \sum_{i=1}^{n} \mathbf{I}_{\{Y_i = \ell\}} \big(X_i - \bar{X}^{(n)}\big) \otimes \big(X_i - \bar{X}^{(n)}\big) \quad \text{and} \quad \hat{\Theta}_{j\ell} = \delta_{j\ell} \hat{p}_j \hat{V}_n^{(j)} + \hat{p}_j \hat{p}_\ell \big(\hat{V}_n - \hat{V}_n^{(j)} - \hat{V}_n^{(\ell)}\big). \tag{4}$$

The following theorem shows the consistency of the test for a sufficiently large p:

Theorem 2.2. *Under $\mathcal{H}_1$, there exists $p_0 \in \mathbb{N}^*$ such that, for any $p \geqslant p_0$ and any $\alpha \in]0, 1[$, we have*:

$$\lim_{n \to +\infty} P\big(\mathbb{F}_{\mathcal{Q}}\big(n\widehat{T}_n(p)\big) > 1 - \alpha\big) = 1.$$

Proof. Since $\|\tau_\ell\|^2 = \sum_{i=1}^{+\infty} \langle \tau_\ell, u_i \rangle^2$ then, under $\mathcal{H}_1$, there exists $(\ell_0, p_0) \in F \times \mathbb{N}^*$ such that $\langle \tau_{\ell_0}, u_{p_0} \rangle^2 > 0$. Therefore, we have for any $p \geqslant p_0$, $T(p) \geqslant \lambda_{p_0}^{-1} p_{\ell_0} \langle \tau_{\ell_0}, u_{p_0} \rangle^2 > 0$. For $p \geqslant p_0$, let us consider a real ε such that $0 < \varepsilon < T(p)$. From the inequality

$$P\big(\big|\widehat{T}_n(p) - T(p)\big| < \varepsilon\big) \leqslant P\big(\widehat{T}_n(p) > T(p) - \varepsilon\big)$$

and from the convergence in probability of $\widehat{T}_n(p)$ to $T(p)$ as $n \to +\infty$, we deduce that

$$\lim_{n \to +\infty} P\big(\widehat{T}_n(p) > T(p) - \varepsilon\big) = 1. \tag{5}$$

Let n_0 be an integer such that $\forall n \geqslant n_0,\ n^{-1} \mathbb{F}_{\mathcal{Q}}^{-1}(1 - \alpha) < T(p) - \varepsilon$. Then, for all $n \geqslant n_0$, we have

$$P\big(\widehat{T}_n(p) > T(p) - \varepsilon\big) \leqslant P\big(\widehat{T}_n(p) > n^{-1} \mathbb{F}_{\mathcal{Q}}^{-1}(1 - \alpha)\big)$$

and the required result is obtained from (5) and this later inequality. □

3. Application to functional variables

Here we show how the test introduced in the previous section can be applied in order to deal with functional variables, so introducing a new approach for testing for homogeneity of means for such variables. Let $X = (X(t))_{t\in[0,1]}$ be a random variable valued into $L^2([0,1])$ and Y be as previously defined. We consider an i.i.d. sample $\{(X_i, Y_i)\}_{1\leqslant i\leqslant n}$ of (X, Y), assuming that each function X_i is observed on fine grids of points $t_1, \dots, t_L$ satisfying $0 \leqslant t_1 < t_2 < \cdots < t_L \leqslant 1$. This leads to the $n \times L$ matrix $\mathbb{X} = (X_i(t_j))_{1\leqslant i\leqslant n,\, 1\leqslant j\leqslant L}$ that contains all observations of X. For showing how the previous test for $\mathcal{H}_0$ can be applied here, we just have to explain how to obtain the matrices $\hat{\Gamma}$ and $\hat{\Sigma}$ in practice. First note that, the $\hat{\lambda}_i$'s and the scores $\hat{c}_i(k) = \langle X_i, \hat{u}_k\rangle$ are output of the function pca.fd that is available in the R package fda, and which require the matrix $\mathbb{X}$ as argument. So, by using this function the matrix $\hat{\Gamma}$ can be easily computed. Moreover, it is easy to check that $\langle \hat{u}_i, \hat{V}_n \hat{u}_k\rangle = \hat{\beta}^n_{ik}$ and $\langle \hat{u}_i, \hat{V}^{(j)}_n \hat{u}_k\rangle = \hat{\beta}^n_{ik|j}$, where

$$\hat{\beta}^n_{ik} = \frac{1}{n}\sum_{r=1}^{n}\big(\hat{c}_r(i) - \bar{c}(i)\big)\big(\hat{c}_r(k) - \bar{c}(k)\big) \quad \text{and} \quad \hat{\beta}^n_{ik|j} = \frac{1}{\hat{n}_j}\sum_{r=1}^{n}\mathbf{I}_{\{Y_r=j\}}\big(\hat{c}_r(i) - \bar{c}(i)\big)\big(\hat{c}_r(k) - \bar{c}(k)\big),$$

with $\bar{c}(i) = n^{-1}\sum_{r=1}^{n}\hat{c}_r(i)$. Therefore, from (4) we deduce that $\hat{\Sigma}$ is constructed as in (3) with entries given by:

$$\hat{\sigma}_{ijk\ell} = \delta_{j\ell}\hat{p}_j\hat{\beta}^n_{ik|j} + \hat{p}_j\hat{p}_\ell\big(\hat{\beta}^n_{ik} - \hat{\beta}^n_{ik|j} - \hat{\beta}^n_{ik|\ell}\big). \tag{6}$$

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References

[1] P. Billingsley, Convergence of Probability Measures, Wiley, 1968.
[2] C. Bugli, P. Lambert, Functional ANOVA with random functional effects: An application to event-related potentials modelling for electroencephalograms analysis, Statistics in Medicine 25 (2006) 3718–3739.
[3] H. Cardot, F. Ferraty, A. Mas, P. Sarda, Testing hypotheses in the linear functional model, Scand. J. Statist. 30 (2003) 241–255.
[4] H. Cardot, A. Goia, P. Sarda, Testing for no effect in functional linear regression models, some computational approaches, Commun. Stat. Simulation Comput. 33 (2004) 179–199.
[5] A. Cuevas, M. Febrero, R. Fraiman, An ANOVA test for functional data, Comput. Statist. Data Anal. 47 (2004) 111–122.
[6] A. Cuevas, R. Fraiman, On depth measures and dual statistics. A methodology for dealing with general data, J. Multivariate Anal. 100 (2009) 753–766.
[7] S. Dossou-Gbete, A. Pousse, Asymptotic study of eigenelements of a sequence of random selfadjoint operators, Statistics 22 (1991) 479–491.
[8] J. Fan, S.K. Lin, Test of significance when data are curves, J. Amer. Statist. Assoc. 93 (1998) 1007–1021.
[9] F. Ferraty, Ph. Vieu, Nonparametric Functional Data Analysis: Theory and Practice, Springer, 2006.
[10] L. Ferré, A.F. Yao, Functional sliced inverse regression analysis, Statistics 37 (2003) 475–488.
[11] P. Kokoszka, I. Maslova, J. Sojka, L. Zhu, Testing for lack of dependence in the functional linear model, Can. J. Statist. 36 (2008) 1–16.
[12] A. Mas, Testing for the mean of random curves: A penalization approach, Stat. Inference Stoch. Process. 10 (2007) 147–163.
[13] A.M. Mathai, S.B. Provost, Quadratic Forms in Random Variables: Theory and Applications, Dekker, 1992.
[14] J.O. Ramsay, B.W. Silverman, Functional Data Analysis, Springer, 2005.