

Algebraic Geometry

The primitive cohomology lattice of a complete intersection

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Abstract

We describe the primitive cohomology lattice of a smooth even-dimensional complete intersection in projective space. *To cite this article: A. Beauville, C. R. Acad. Sci. Paris, Ser. I 347 (2009).*

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Résumé

La cohomologie primitive d'une intersection complète. Nous décrivons le réseau de cohomologie primitive d'une intersection complète lisse de dimension paire dans l'espace projectif. *Pour citer cet article : A. Beauville, C. R. Acad. Sci. Paris, Ser. I 347 (2009).*

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1. Introduction

Let X be a smooth complete intersection of degree d and even dimension n in projective space. We describe in this note the lattice structure of the primitive cohomology $H^n(X, \mathbf{Z})_0$. Excluding the cubic surface and the intersection of two quadrics, we find

$$H^n(X, \mathbf{Z})_0 = A_{d-1} \oplus pE_8(\pm 1) \oplus qU \quad \text{or} \quad \langle -d \rangle \oplus p'E_8(\pm 1) \oplus q'U$$

where the numbers p, q, p', q' and the sign attributed to E_8 depend on the multidegree and dimension of X — see Theorem 4 for a precise statement. The proof is an easy consequence of classical facts on unimodular lattices together with the Hirzebruch formula for the Hodge numbers of X .

We warn the reader that there are many ways to write an indefinite lattice as an orthogonal sum of indecomposable ones; for instance, when $8|d$, both decompositions above hold. Still it might be useful to have a (semi-) uniform expression for this lattice. Related results, with a different point of view, appear in [3].

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2. Unimodular lattices

As usual we denote by U the rank 2 hyperbolic lattice, and by $\langle d \rangle$ the lattice $\mathbf{Z}e$ with $e^2 = d$. If L is a lattice, $L(-1)$ denotes the \mathbf{Z} -module L with the form $x \mapsto -x^2$; if n is a negative number, we put $nL := |n|L(-1)$.

Let L be an odd unimodular lattice. A primitive vector $h \in L$ is said to be *characteristic* if $h \cdot x \equiv x^2 \pmod{2}$ for all $x \in L$; this is equivalent to saying that the orthogonal lattice h^\perp is even [3, Lemma 3.3].

Proposition 1. *Let L be a unimodular lattice, of signature (b^+, b^-) , with $b^+, b^- \geq 2$; let h be a primitive vector in L of square $d > 0$, such that h^\perp is even. Put $s := b^+ - b^-$, $t = \min(b^+, b^-)$, $u = \min(b^-, b^+ - d)$.*

- 1) *If L is even or $8|d$ we have $h^\perp = \langle -d \rangle \oplus \frac{s}{8}E_8 \oplus (t - 1)U$.*
- 2) *If L is odd and $d \leq b^+$, we have $h^\perp = A_{d-1} \oplus \frac{s-d}{8}E_8 \oplus uU$.*

Proof. A classical result of Wall [6] tells us that h is equivalent under $O(L)$ to any primitive vector v of square d , provided v is characteristic if so is h . If L is even, we choose a hyperbolic plane $U \subset L$ with a hyperbolic basis (e, f) , and we put $v = e + \frac{d}{2}f$; then $v^\perp = \mathbf{Z}(e - \frac{d}{2}f) \oplus U^\perp$, and U^\perp is an indefinite unimodular lattice, hence of the form $pE_8(\pm 1) \oplus qU$. Computing b^+ and b^- we find the above expressions for p and q .

Consider now the case when L is odd. We first observe that since h is characteristic, we have $d = h^2 \equiv s \pmod{8}$ [5, V, Theorem 2]. Let

$$L' := \left(\bigoplus_{i \leq d} \mathbf{Z}e_i \right) \oplus \frac{s-d}{8}E_8 \oplus uU \quad \text{with} \quad e_1^2 = \dots = e_d^2 = 1.$$

L' is odd, indefinite and has the same signature as L , hence is isometric to L . We put $v = e_1 + \dots + e_d$. The orthogonal of v in $\bigoplus \mathbf{Z}e_i$ is the root lattice A_{d-1} . By Wall’s theorem h^\perp is isometric to $v^\perp = A_{d-1} \oplus \frac{s-d}{8}E_8 \oplus uU$.

Suppose moreover that 8 divides d , so that $8|s$. Then L is isomorphic to $\mathbf{Z}e \oplus \mathbf{Z}f \oplus \frac{s}{8}E_8 \oplus (t - 1)U$, with $e^2 = 1, f^2 = -1$. Taking $v = (\frac{d}{4} + 1)e + (\frac{d}{4} - 1)f$ gives the result. \square

Remark. Since the signature of h^\perp is $(b^+ - 1, b^-)$, the condition $d \leq b^+$ is necessary in order that h^\perp contains A_{d-1} .

3. Complete intersections

We will check that the hypotheses of the proposition hold for the cohomology of complete intersections; the only non-trivial point is the inequality $d \leq b^+$.

We will use the notations of [1]. Let $\mathbf{d} = (d_1, \dots, d_c)$ be a sequence of positive integers. We denote by $V_n(\mathbf{d})$ a smooth complete intersection of multidegree \mathbf{d} in \mathbf{P}^{n+c} . We put

$$h^{p,q}(\mathbf{d}) = \dim H^{p,q}(V_{p+q}(\mathbf{d})) \quad \text{and} \quad h_0^{p,q}(\mathbf{d}) = h^{p,q}(\mathbf{d}) - \delta_{p,q}.$$

Lemma 2. $h^{p+1,q+1}(\mathbf{d}) \geq h^{p,q}(\mathbf{d})$.

Proof. Following [1] we introduce the formal generating series

$$H(\mathbf{d}) = \sum_{p,q \geq 0} h_0^{p,q}(\mathbf{d}) y^p z^q \in \mathbf{Z}[[y, z]];$$

we define a partial order on $\mathbf{Z}[[y, z]]$ by writing $P \geq Q$ if $P - Q$ has non-negative coefficients. The assertion of the lemma is equivalent to $H(\mathbf{d}) \geq yzH(\mathbf{d})$. The set \mathcal{P} of formal series in $\mathbf{Z}[[y, z]]$ with this property is stable under addition and multiplication by any $P \geq 0$ in $\mathbf{Z}[[y, z]]$. The formula

$$H(d_1, \dots, d_c) = \sum_{\substack{P \subset [1, d] \\ P \neq \emptyset}} [(1+y)(1+z)]^{|P|-1} \prod_{i \in P} H(d_i)$$

[1, Corollary 2.4(ii)] shows that it is enough to prove that $H(d)$ is in \mathcal{P} .

By [1, Corollary 2.4(i)], we have $H(d) = \frac{P}{1-Q}$ with

$$P(y, z) = \sum_{i, j \geq 0} \binom{d-1}{i+j+1} y^i z^j \quad \text{and} \quad Q(y, z) = \sum_{i, j \geq 1} \binom{d}{i+j} y^i z^j.$$

Since $Q \geq yz$, we get $\frac{1-yz}{1-Q} = 1 + \frac{Q-yz}{1-Q} \geq 0$, hence $(1-yz)H \geq 0$. \square

Lemma 3. *Let $d = d_1 \cdots d_c$. We have:*

- a) $h^{p,p}(\mathbf{d}) \geq d$;
- b) $2h^{p+1,p-1}(\mathbf{d}) + 1 \geq d$, except in the following cases:
 - $\mathbf{d} = (2), (2, 2)$;
 - $p = 1, \mathbf{d} = (3), (4), (2, 3), (2, 2, 2), (2, 2, 2, 2)$;
 - $p = 2, \mathbf{d} = (2, 2, 2)$.

Proof. We first prove b) in the case $p = 1$. Then $V_2(\mathbf{d})$ is a surface $S \subset \mathbf{P}^{c+2}$. The canonical bundle K_S is $\mathcal{O}_S(e)$, with $e := d_1 + \cdots + d_c - c - 3$; therefore $K_S^2 = e^2 d$. The cases with $e \leq 0$ are excluded, so we assume $e \geq 1$. Then the index $K_S^2 - 8\chi(\mathcal{O}_S)$ of the intersection form is negative [4]; if $e \geq 2$, we get $\chi(\mathcal{O}_S) > \frac{d}{2}$, hence $2h^{2,0}(\mathbf{d}) + 1 \geq d$.

If $e = 1$, we have $K_S = \mathcal{O}_S(1)$ hence $p_g = c + 3$. The possibilities for \mathbf{d} are $(5), (2, 4), (3, 3)$ and $(2, 2, 3)$; we have $2(c + 3) + 1 \geq d$ in each case.

Since the index is negative, we have $h^{1,1}(\mathbf{d}) > 2h^{2,0}(\mathbf{d}) + 1$; this implies that a) holds (for $p = 1$) except perhaps for $\mathbf{d} = (3), (2, 2), (4), (2, 3), (2, 2, 2)$. But the corresponding $h^{1,1}$ is 7, 6, 20, 20, 20, which is always $> d$.

Now assume $p \geq 2$. a) follows from the previous case and Lemma 2; similarly it suffices to check b) for the values of \mathbf{d} excluded in the case $p = 1$. Using the above formulas we find

$$h^{3,1}(3) = 1, \quad h^{3,1}(4) = 21, \quad h^{3,1}(2, 3) = 8, \quad h^{3,1}(2, 2, 2, 2) = 27, \quad h^{4,2}(2, 2, 2) = 6,$$

so that $2h^{p+1,p-1}(\mathbf{d}) + 1 \geq d$ for $p \geq 2$ in the three first cases and for $p \geq 3$ in the last one. \square

Theorem 4. *Let X be a smooth even-dimensional complete intersection in \mathbf{P}^{n+c} , of multidegree $\mathbf{d} = (d_1, \dots, d_c)$. Let $d := d_1 \cdots d_c$ be the degree of X , and let e be the number of integers d_i which are even.*

Let (b^+, b^-) be the signature of the intersection form on $H^n(X, \mathbf{Z})$; we put

$$s = b^+ - b^-, \quad t = \min(b^+, b^-), \quad u = \min(b^+ - d, b^-).$$

We assume $\mathbf{d} \neq (2, 2)$ and $\mathbf{d} \neq (3), (2, 2, 2, 2)$ when $n = 2$. Then:

- $H^n(X, \mathbf{Z})_0 = \langle -d \rangle \oplus \frac{s}{8} E_8 \oplus (t-1)U$ if $(\frac{n}{2} + e)$ is even;
- $H^n(X, \mathbf{Z})_0 = A_{d-1} \oplus \frac{s-d}{8} E_8 \oplus uU$ if $(\frac{n}{2} + e)$ is odd.

For a hypersurface, for instance, we find a lattice of the form $A_{d-1} \oplus pE_8 \oplus qU$ except if d is even and $n \equiv 2 \pmod{4}$.

Proof. We apply Proposition 1 with $L = H^n(X, \mathbf{Z})$. We take for h the class of a linear section of codimension $\frac{n}{2}$, so that $h^2 = d$.

By [3, Theorem 2.1 and Corollary 2.2], we know that

- h is primitive;
- h^\perp is even;

- L is even or odd according to the parity of $\binom{\frac{n}{2}+e}{e}$.

To apply the proposition we only need the inequalities $b^+ \geq d$ and $b^- \geq 2$. Note that the statement of the theorem holds trivially for $\mathbf{d} = (2)$, so we may assume $d \geq 3$. Let us write $n = 4k + 2\varepsilon$, with $\varepsilon \in \{0, 1\}$. By Hodge theory we have

$$b^+ = \sum_{\substack{p+q=n \\ p \text{ even}}} h^{p,q} + \varepsilon, \quad b^- = \sum_{\substack{p+q=n \\ p \text{ odd}}} h^{p,q} - \varepsilon;$$

when the inequalities a) and b) of Lemma 3 hold this implies $b^+ \geq d$ and $b^- \geq 2$, so Proposition 1 gives the result.

In the remaining cases $p = 1$, $\mathbf{d} = (3), (2, 3), (2, 2, 2)$ and $p = 2$, $\mathbf{d} = (2, 2, 2)$, the lattice L is even and we have $b^+, b^- \geq 2$, so Proposition 1 still applies. \square

Remark. The two first exceptions mentioned in the theorem are well-known [2, Proposition 5.2]: we have $H^2(X, \mathbf{Z})_o = E_6$ for a cubic surface, and $H^n(X, \mathbf{Z})_o = D_{n+3}$ for a n -dimensional intersection of two quadrics. For an intersection of 4 quadrics in \mathbf{P}^6 , we have $d = 16$, hence by Proposition 1

$$H^2(X, \mathbf{Z})_o = \langle -16 \rangle \oplus 6E_8(-1) \oplus 15U.$$

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