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C. R. Acad. Sci. Paris, Ser. I 347 (2009) 1351–1354
COMPTES RENDUS MATHEMATIQUE

Mathematical Analysis

Flatness of distributions vanishing on infinitely many hyperplanes

Jan Boman

Department of Mathematics, Stockholm University, SE 10691, Stockholm, Sweden

Received 7 April 2009; accepted after revision 28 October 2009

Available online 10 November 2009

Presented by Jean-Michel Bony

Abstract

Let $\{L_k\}_{k=1}^{\infty}$ be a family of hyperplanes in $\mathbf{R}^n$ and let L_0 be a limiting hyperplane of $\{L_k\}$. Let u be a distribution that satisfies a natural wave front condition and has vanishing restrictions to L_k for all $k \geqslant 1$. Then u must be flat at L_0. ***To cite this article: J. Boman, C. R. Acad. Sci. Paris, Ser. I 347 (2009).***

Résumé

Platitude des distributions s'annulant sur une infinité d'hyperplans. Soit $\{L_k\}_{k=1}^{\infty}$ une famille d'hyperplans dans $\mathbf{R}^n$ et soit L_0 un hyperplan limite de $\{L_k\}$. Si u est une distribution satisfaisant à une condition naturelle portant sur le front d'onde et qui s'annule sur L_k pour tout $k \geqslant 1$, alors u est plate sur L_0. ***Pour citer cet article : J. Boman, C. R. Acad. Sci. Paris, Ser. I 347 (2009).***

1. Introduction

Let $\mathcal{L}$ be an infinite family of distinct hyperplanes L in $\mathbf{R}^n$ with limit point (in the natural topology on the n-dimensional manifold of hyperplanes) L_0, and let U be an open set in $\mathbf{R}^n$ intersecting L_0. Let u be an infinitely differentiable function vanishing on $L \cap U$ for all $L \in \mathcal{L}$. Then it is easy to see that u must be flat on $L_0 \cap U$ in the sense that the derivatives of u of all orders vanish on L_0. To prove this, assume that some derivative of order m of u is different from zero at $x^0 \in L_0$ and that all derivatives of order $< m$ vanish at x^0. We may assume that x^0 is the origin in $\mathbf{R}^n$. Then $u(x) = p(x)(1 + \mathcal{O}(|x|))$ as $|x| \to 0$, where $p(x)$ is a non-zero homogeneous polynomial of degree m. Then the restriction of u to L must be non-identically zero for any hyperplane L with sufficiently small distance to the origin. This proves the statement.

The purpose of this note is to show that a similar statement is true for distributions u, provided that

$$WF(u) \cap N^*(L_0 \cap U) = \emptyset, \tag{1}$$

a condition which is needed for the restriction of u to L_0 to be well defined [4, Corollary 8.2.7]. Here $N^*(L_0)$ denotes the conormal manifold to L_0, i.e., the set of (x, ξ) where $x \in L_0$ and ξ is conormal to L_0 at x, and $WF(u)$ is the wave

E-mail address: jabo@math.su.se.
URL: http://www.math.su.se/~jabo.

1631-073X/$ – see front matter
doi:10.1016/j.crma.2009.10.028

front set of u. By definition $(x^0, \xi^0) \notin WF(u)$ if there exists a function $\psi \in C_0^\infty$ such that $\psi(x^0) \neq 0$ and a conic neighborhood Γ of ξ^0 such that $\widehat{\psi u}(\xi)$ is rapidly decaying in Γ in the sense that

$$\left|\widehat{\psi u}(\xi)\right| \leqslant C_m/|\xi|^m, \quad \xi \in \Gamma,\ m = 1, 2, \ldots. \tag{2}$$

Assume L_0 is the hyperplane $x_n = 0$. If (1) holds and $x^0 \in L_0 \cap U$ we can choose $\psi \in C_0^\infty(U)$ with $\psi(x^0) \neq 0$ such that (2) holds with $\Gamma = \{\xi = (\xi', \xi_n);\ |\xi'| < \delta|\xi_n|\}$ for some $\delta > 0$. If $\varphi(x')$ is a test function in $C_0^\infty(\mathbf{R}^{n-1})$ defined on L_0, then the action of the restriction $\psi u|_{L_0}$ on the test function φ can be defined by

$$\langle \psi u|_{L_0}, \varphi\rangle = (2\pi)^{-n} \int\limits_{\mathbf{R}^n} \widehat{\psi u}(\xi', \xi_n)\hat{\varphi}(-\xi')\, \mathrm{d}\xi.$$

Note that the integral must be absolutely convergent because of (2).

The space of (Schwartz) distributions on the open set U is denoted $\mathcal{D}'(U)$.

Theorem 1. *Let $\{L_k\}_{k=1}^\infty$ be an infinite family of (distinct) hyperplanes in $\mathbf{R}^n$, $n \geqslant 2$, and let $L_0 = \lim_{k\to\infty} L_k$ (in the topology of the manifold of hyperplanes). Let U be a bounded open subset of $\mathbf{R}^n$. Assume $u \in \mathcal{D}'(U)$ satisfies* (1) *and*

$$\textit{the restriction } u|_{L_k \cap U} \textit{ vanishes for all } k \geqslant 1. \tag{3}$$

Then u is flat in the set $L_0 \cap (U \setminus F)$, where F is the (possibly empty) affine subspace of L_0 of codimension $\geqslant 1$

$$F = \bigcup_{m=1}^\infty \bigcap_{k=m}^\infty (L_k \cap L_0), \tag{4}$$

that is, for every partial derivative ∂^α of u the restriction of $\partial^\alpha u$ to $L_0 \cap (U \setminus F)$ vanishes.

Note that the wave front condition (1) is satisfied for all hyperplanes L sufficiently close to L_0, since the wave front set $WF(u)$ is closed. Since $WF(\partial^\alpha u) \subset WF(u)$ for any partial derivative and any distribution u, the same is true for all partial derivatives $\partial^\alpha u$. Note also that the set $F_m = \bigcap_{k=m}^\infty (L_k \cap L_0)$ is a (possibly empty) affine subspace of L_0 of codimension $\geqslant 1$ for every m (we may assume that $L_k \neq L_0$ for all k), and the sequence F_m is increasing, so it is clear that F is an affine subspace as stated in the theorem. F can be described as the set of all points that are contained in all except finitely many of the sets $L_0 \cap L_k$.

The fact that the exceptional set F may occur can be seen from the following example. Let u be the distribution on $\mathbf{R}^2$ defined by $u(x_1, x_2) = x_2\delta_0(x_1)$. Then $WF(u)$ is equal to the conormal of the line $x_1 = 0$, that is, $WF(u) = \{(0, x_2; \xi_1, 0);\ x_2 \in \mathbf{R},\ \xi_1 \neq 0\}$. The restriction of u to any of the lines $L_k = \{x \in \mathbf{R}^2;\ x_2 = x_1/k\}$, $k = 1, 2, \ldots,$ is well defined and vanishes, but the restriction of $\partial_{x_2} u = \delta_0(x_1)$ to $L_0 = \{x \in \mathbf{R}^2;\ x_2 = 0\}$ is $\delta_0(x_1)$, so u is not flat on all of L_0 but only on $L_0 \setminus F$ where $F = \{(0, 0)\}$.

The assertion of Theorem 1 is in fact valid also if $n = 1$, because then condition (1) means that u is C^∞ in some neighborhood of the point L_0, and a smooth function vanishing at an infinite sequence of points must be flat at a limit point of that sequence; note that a hyperplane (affine submanifold of codimension 1) means a point in this case.

By our vanishing theorem for microlocally real analytic flat distributions [2], a distribution u that satisfies the *analytic* wave front condition

$$WF_A(u) \cap N^*(L_0 \cap U) = \emptyset \tag{5}$$

and is flat on $L_0 \cap U$ must vanish in some neighborhood of $L_0 \cap U$. (For the definition of the analytic wave front set, $WF_A(u)$, see [4].) Combining this fact with Theorem 1 we obtain the following extension of the familiar fact that a real analytic function of one variable that vanishes at infinitely many points with a finite limit point must vanish identically:

Corollary 2. *Let L_k, $k = 1, 2, \ldots,$ be an infinite sequence of distinct hyperplanes in $\mathbf{R}^n$ and let $L_0 = \lim_{k\to\infty} L_k$ as in Theorem* 1. *Let U be a bounded open subset of $\mathbf{R}^n$. Assume $u \in \mathcal{D}'(U)$ satisfies* (5) *and* (3). *Then $u = 0$ in some neighborhood of $L_0 \cap (U \setminus F)$, where F is the set defined by* (4).

In the recent article [3] we applied this corollary to give a new proof of a uniqueness result for a ray transform [1].

2. Proof of Theorem 1

Since F is closed it is sufficient to prove that u is flat on $L_0 \cap U_1$ for some open neighborhood U_1 of an arbitrary point of $U \setminus F$. Thus from now on we denote U_1 by U and assume that $\overline{U} \cap F = \emptyset$.

We denote the coordinates in $\mathbf{R}^n$ by (x, y) where $x \in \mathbf{R}^{n-1}$, $y \in \mathbf{R}$, and the dual coordinates by (ξ, η). We may assume that L_0 is the plane $y = 0$. Let $(x^0, 0) \in U \cap L_0$ and choose a neighborhood $V \subset U$ of $(x^0, 0)$, a conic neighborhood $\Gamma = \{(\xi, \eta);\ |\xi| < \delta|\eta|\}$ of the conormal $(0, \pm 1)$ to L_0, and a function $\psi \in C_0^\infty(V)$ such that $\psi(x^0, 0) \neq 0$ and $\widehat{\psi u}(\xi, \eta)$ is rapidly decaying in the cone Γ. From now on we shall denote ψu by u. We may assume that u is a real-valued distribution in the sense that $\langle u, \varphi\rangle$ is real for all real-valued test functions φ. Denote by $u|_{L_{a,b}}$ the restriction of u to the plane $y = a \cdot x + b$, where $a \in \mathbf{R}^{n-1}$ and $b \in \mathbf{R}$. If $|a|$ and $|b|$ are sufficiently small this restriction is well defined, and its action on a test function $\varphi \in C_0^\infty(\mathbf{R}^{n-1})$ can be written

$$\langle u|_{L_{a,b}}, \varphi\rangle = (2\pi)^{-n} \iint \hat{u}(\xi, \eta)\hat{\varphi}(-\xi - \eta a)e^{ib\eta}\, \mathrm{d}\xi\, \mathrm{d}\eta.$$

The fact that $\hat{u}$ has at most polynomial growth in the ξ-variable and is rapidly decaying in the cone $|\xi| < \delta|\eta|$ implies that the integral is absolutely convergent if $|a|$ is sufficiently small. For real-valued $\varphi \in C_0^\infty(\mathbf{R}^{n-1})$ we set

$$\begin{aligned}\rho_{a,b}(s) = \rho_{\varphi,a,b}(s) &= \langle u|_{L_{sa,sb}}, \varphi\rangle \\ &= (2\pi)^{-n} \iint \hat{u}(\xi, \eta)\hat{\varphi}(-\xi - s\eta a)e^{isb\eta}\, \mathrm{d}\xi\, \mathrm{d}\eta, \quad s \in \mathbf{R}.\end{aligned} \tag{6}$$

Then $\rho_{a,b}(0) = \langle u|_{L_0}, \varphi\rangle$ and $\rho_{a,b}(1) = \langle u|_{L_{a,b}}, \varphi\rangle$. It is clear that $\rho_{a,b} \in C^\infty$ in $(-1, 1)$, if $|a| < \delta/2$. Differentiating m times with respect to s we obtain

$$\rho_{a,b}^{(m)}(s) = (2\pi)^{-n} \iint \hat{u}(\xi, \eta)\eta^m(-a \cdot \nabla + ib)^m \hat{\varphi}(-\xi - s\eta a)e^{isb\eta}\, \mathrm{d}\xi\, \mathrm{d}\eta \tag{7}$$

and

$$\rho_{a,b}^{(m)}(0) = \langle \partial_y^m u|_{L_0}, \psi\rangle, \quad \text{where } \psi(x) = (a \cdot x + b)^m \varphi(x).$$

Assume that $\mathcal{L} = \{L_k\}$ where L_k is the plane $y = a_k \cdot x + b_k$. We now claim that for every m there exists $s_0 = s_0(\varphi, k, m) \in (0, 1]$ such that

$$\rho_{a_k,b_k}^{(m)}(s_0) = 0, \quad \text{for all } k \text{ and all } \varphi. \tag{P_m}$$

We shall prove this by induction over m. For $m = 0$ (P_m) is true with $s_0 = 1$ by the assumption that $u|_{L_{a_k,b_k}} = 0$ for all k. Let m be arbitrary and assume that (P_m) holds. Using the expression (7) for $\rho_{a,b}^{(m)}$ in (P_m), dividing by $(|a_k|^2 + b_k^2)^{1/2}$, passing to a subsequence such that $(a_k, b_k)/(|a_k|^2 + b_k^2)^{1/2}$ converges to a limit (a_0, b_0), and letting k tend to infinity we obtain

$$0 = \iint \hat{u}(\xi, \eta)\eta^m(-a_0 \cdot \nabla + ib_0)^m \hat{\varphi}(-\xi)\, \mathrm{d}\xi\, \mathrm{d}\eta = \langle \partial_y^m u|_{L_0}, (a_0 \cdot x + b_0)^m \varphi\rangle.$$

Our assumption that $\overline{U} \cap F = \emptyset$ implies that $a_0 \cdot x + b_0 \neq 0$ on $L_0 \cap U$. Since φ is arbitrary it follows that $\partial_y^m u|_{L_0} = 0$ in U, and hence $\rho_{a,b}^{(m)}(0) = 0$ for every φ and every a and b. For an arbitrary k we now use the induction assumption (P_m) once more together with Rolle's theorem and obtain

$$0 = \rho_{a_k,b_k}^{(m)}(s_0) - \rho_{a_k,b_k}^{(m)}(0) = \rho_{a_k,b_k}^{(m+1)}(s_1)$$

for some $s_1 \in (0, s_0)$, which proves (P_{m+1}), and hence proves (P_m) for every m. By (6) we know that $\rho_{a,b}$ is real-valued, so the application of Rolle's theorem is appropriate. Letting k tend to infinity and using (7) now gives

$$0 = \iint \hat{u}(\xi, \eta)\eta^m \hat{\varphi}(-\xi)\, \mathrm{d}\xi\, \mathrm{d}\eta, \tag{8}$$

that is, $\partial_y^m u|_{L_0} = 0$ for every m.

It remains to show that an arbitrary mixed derivative $\partial_x^\alpha \partial_y^m u$, where $\alpha = (\alpha_1, \ldots, \alpha_{n-1})$, must have vanishing restriction to L_0. By definition this means that

$$\iint \hat{u}(\xi, \eta)\xi^\alpha \eta^m \hat{\varphi}(-\xi)\, \mathrm{d}\xi\, \mathrm{d}\eta = 0, \quad \varphi \in C_0^\infty\big(\mathbf{R}^{n-1}\big). \tag{9}$$

Replacing φ in (8) with $\partial_x^\alpha \varphi$ gives (9) and completes the proof.

Acknowledgement

The author thanks Professor Jan-Erik Björk for stimulating discussions and valuable suggestions leading to an improvement of the exposition.

References

[1] C. Béslisle, J.-C. Massé, T. Ransford, When is a probability measure determined by infinitely many projections? Ann. Probab. 25 (1997) 767–786.
[2] J. Boman, A local vanishing theorem for distributions, C. R. Acad. Sci. Paris, Ser. I 315 (1992) 1231–1234.
[3] J. Boman, Unique continuation of microlocally analytic distributions and injectivity theorems for the ray transform, Inverse Probl. Imaging, in press.
[4] L. Hörmander, The Analysis of Linear Partial Differential Operators, vol. 1, Springer-Verlag, Berlin, Heidelberg, New York, Tokyo, 1983.