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# The Goursat problem for the Einstein–Yang–Mills–Higgs system in weighted Sobolev spaces

## *Problème de Goursat pour les équations d'Einstein–Yang–Mills–Higgs dans les espaces de Sobolev à poids*

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A B S T R A C T

We establish original Moser estimates to clarify and complete previous works of Christodoulou and Müller zum Hagen concerning local existence and uniqueness results for the Goursat problem associated to second order quasilinear hyperbolic systems. As an application we locally solve, in some weighted Sobolev spaces, the Goursat problem for the Einstein–Yang–Mills–Higgs system using harmonic and Lorentz gauges.


R É S U M É

Nous établissons des estimations de Moser originales pour clarifier et compléter des travaux antérieurs de Christodoulou et Müller zum Hagen portant sur des résultats locaux d'existence et d'unicité pour le problème de Goursat associé aux systèmes hyperboliques quasilinéaires du second ordre. Comme application nous résolvons localement, dans des espaces de Sobolev à poids, le problème de Goursat pour le système Einstein–Yang–Mills–Higgs en jauges harmonique et de Lorentz.


### Version française abrégée

L'objet de ce travail est la résolution locale du problème de Goursat (les données initiales sont prescrites sur deux hypersurfaces caractéristiques régulières sécantes) pour le système Einstein–Yang–Mills–Higgs (EYMH) (11) dans des espaces de Sobolev à poids appropriés. Nous traitons à la fois le problème de l'évolution et celui des contraintes initiales. Via le choix des jauges harmoniques et de Lorentz (12), le système réduit des équations EYMH se présente sous forme d'un système hyperbolique quasilinéaire du second ordre (13). Cet état de choses nous a conduit dans un premier temps à résoudre le problème de Goursat pour les systèmes hyperboliques quasilinéaires du second ordre (4), clarifiant et complétant ainsi des travaux antérieurs de Christodoulou et Müller zum Hagen [2,7]. Pour y arriver nous déployons une méthode de point fixe dont les principaux outils sont : les résultats d'existence et d'unicité pour les systèmes hyperboliques linéaires du second ordre obtenus dans [7], les inégalités de type Sobolev établies dans [2,7] et de nouvelles et originales estimations de Moser [6] (lemmes de substitution) qui constituent le supplément qui faisait défaut dans [2,7] et empêchait de résoudre

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clairement le problème de Goursat quasilinéaire. Dans un deuxième temps nous appliquons le résultat ainsi mis à jour au système réduit des équations EYMH (13), ce qui permet de résoudre le problème de l'évolution associé au système EYMH complet (11). La résolution du problème des contraintes associé au système EYMH (11) dans les espaces de Sobolev à poids définis dans [7] se fait par une adaptation heureuse de la méthode mise sur pieds par A.D. Rendall [8] et utilisée par la suite par T. Damour and B.G. Schmidt [3] pour traiter, sous des hypothèses de différentiabilité $C^\infty$, les équations d'Einstein du vide et avec source fluide parfait relativiste ainsi que les équations d'Einstein–Yang–Mills pures. L'utilisation des conditions de jauges harmonique et de Lorentz permet de conclure que la solution du système réduit (13) est également solution du système complet EYMH (11). Au bout du compte nous obtenons une amélioration et une généralisation des résultats antérieurs à l'instar de ceux obtenus dans [2,3,7,8].

## 1. Spaces of functions used and Moser estimates

$L$ denotes a compact domain of $\mathbb{R}^{n+1}$ ($n \geqslant 2$), with a piecewise smooth boundary $\partial L$. For $\omega = 1,2$, define $G^\omega = \{x \in L\colon x^\omega = 0\}$, where $x = (x^a)_{a=1,\dots,n+1}$ is the global canonical coordinates system of $\mathbb{R}^{n+1}$. Assume that $G^1 \cup G^2 \subset \partial L$. Define a time-function $\tau(x)$ by $\tau(x) = x^1 + x^2$. For $s \in \mathbb{N}$ and $t \in (0, T_0]$ where $T_0 = \sup x \in L\tau(x)$, define the following point sets and weighted norms as in [7]:

$$L_t = \{x \in L\colon 0 \leqslant \tau(x) \leqslant t\}, \qquad \Lambda_t = \{x \in L\colon \tau(x) = t\}, \qquad G_t^\omega = \{x \in G^\omega\colon 0 \leqslant \tau(x) \leqslant t\},$$

$$\Gamma_t^\omega = \{x \in G^\omega\colon \tau(x) = t\}, \qquad \|v\|_{H_s(S_t,R)} \equiv |v|_s^{S_t,R} = t^{-\alpha}\left(\sum_{k=0}^{s}\int_{S_t}\left|D_R^k v\right|^2 \mathrm{d}S_t\right)^{\frac{1}{2}},$$

$$\|v\|_{H_s(S_t)} \equiv |v|_s^{S_t} = |v|_s^{S_t,S_t}, \qquad \|v\|_{E_s(S_t)} \equiv |v|_{S_t,s} = \operatorname*{ess\,sup}_{0\leqslant\sigma\leqslant t}|v|_s^{\Sigma_\sigma,S_t}, \quad \text{with } S_t = \bigcup_{0\leqslant\sigma\leqslant t}\Sigma_\sigma,$$

where $\alpha = \frac{1}{2}$ if $S_t \in \{\Lambda_t, G_t^\omega\}$, $\alpha = 1$ if $S_t = L_t$, $\alpha = 0$ if $S_t = \Gamma_t^\omega, \Gamma$. $R$ is a surface (submanifold) of $\mathbb{R}^{n+1}$ such that $S_t \subset R \subset L$, $D_R^k$ are $k$-th order derivatives (in distributional sense) tangent to $R$, $|D_R^k v|$ is the norm of $D_R^k v$ w.r.t. the Kronecker metric $\delta^{ab}$, $D_a = \frac{\partial}{\partial x^a}$, $dS_t$ is the volume element induced on $S_t$ by $dx^1 \cdots dx^{n+1}$.

We also define further norms as follows (see [7]):

$$\|v\|_{\mathbb{H}_s(G_t^\omega)} \equiv \|v\|_s^{G_t^\omega} = \left[\sum_{k=0}^{s-1}\left(\left|D_\omega^k v\right|_{2(s-k)-1}^{G_t^\omega}\right)^2\right]^{\frac{1}{2}}, \qquad \|v\|_{\mathbb{H}_s(L_t)} \equiv \|v\|_s^{L_t} = \left[\left(|v|_s^{L_t}\right)^2 + \sum_{\omega=1}^{2}\left(\|v\|_s^{G_t^\omega}\right)^2\right]^{\frac{1}{2}},$$

$$\|v\|_{\mathbb{E}_s(G_t^\omega)} \equiv \|v\|_{G_t^\omega,s} = \left[\sum_{k=0}^{s-1}\sum\left(\left|D_\omega^k v\right|_{G_t^\omega,2(s-k)-1}\right)^2\right]^{\frac{1}{2}}, \qquad \|v\|_{\mathbb{E}_s(L_t)} \equiv \|v\|_{L_t,s} = \left[\left(|v|_{L_t,s}\right)^2 + \sum_{\omega=1}^{2}\left(\|v\|_{G_t^\omega,s}\right)^2\right]^{\frac{1}{2}}.$$

$C^\infty(L_t)$ denotes the space of restrictions to $L_t$ of functions which are $C^\infty$ on $\mathbb{R}^{n+1}$. $H_s(L_t)$ and $\mathbb{H}_s(L_t)$ are Hilbert spaces and $C^\infty(L_t)$ is dense in $H_s(L_t)$. $E_s(L_t)$ and $\mathbb{E}_s(L_t)$ are Banach spaces. We will also need the following Banach spaces $K_s(G_T^1) = \{v \in E_s(G_t^1)\colon D_2 v \in E_s(G_t^1)\}$, $K_s(G_t^2) = \{v \in E_s(G_t^2)\colon D_1 v \in E_s(G_t^2)\}$ with their respective norms $\|v\|_{K_s(G_T^1)} = |v|_{G_T^1,s} + |D_2 v|_{G_T^1,s}$ and $\|v\|_{K_s(G_T^2)} = |v|_{G_T^2,s} + |D_1 v|_{G_T^2,s}$. Let $W$ be an open subset of $\mathbb{R}^l$, $l \in \mathbb{N}^*$, with $\mathbb{N}^* = \mathbb{N}\backslash\{0\}$. $C_b^k(L_t \times W)$ denotes the space of functions $f : L_t \times W \to \mathbb{R}$ such that $D^i f$ exists (in the usual sense) for all $i = 0,\dots,k$ and are continuous and bounded on $L_t \times W$. $C_b^k(L_t \times W)$ is endowed with the norm $\|f\|_{C_b^k(L_t\times W)} = \sup_{(x,w)\in L_t\times W, |\alpha+\beta|\leqslant k}|D_x^\alpha D_w^\beta f(x,w)|$. The following Moser estimates play a crucial role in the resolution of the quasilinear Goursat problem (initial data are assigned on two intersecting smooth null hypersurfaces). The detailed proofs are provided in [9] by adapting the tools of [4,5].

**Theorem 1.1.** *Let $u_i \in \mathbb{E}_s(L_t)$, $i = 1,\dots,l$; $W$ an open subset of $\mathbb{R}^l$; $f : L_t \times W \to \mathbb{R}$ such that $f \in C_b^{2m-1}(L_t \times W)$; $u : L_t \to W$ a mapping $u(x) = (u_1(x),\dots,u_l(x))$. We assume that $1 \leqslant m \leqslant s$, $n < 2s$. Then the function $f(x,u(x))$ of $x$ satisfies the following inequality*

$$\left\|f(x,u(x))\right\|_{L_t,m} \leqslant c_t(m,s)\|f\|_{C_b^{2m-1}(L_t\times W)}\left[1 + \|u\|_{L_t,s}\right]^{2m-1}, \tag{1}$$

*$c_t(m,s)$ being a non-decreasing function of $t$, depending also on $m$ and $s$.*

**Theorem 1.2.** (i) *Let $u_i, v_i \in \mathbb{E}_s(L_t)$, $i = 1,\dots,l$; $u, v : L_t \to W$ two mappings, $u(x) = (u_1(x),\dots,u_l(x))$ and $v(x) = (v_1(x),\dots,v_l(x))$; $f : L_t \times W \to \mathbb{R}$ such that $f \in C_b^{2m-3}(L_t \times W)$, $f(x,0) = 0$ and $D_{u_i} f \equiv \frac{\partial f}{\partial u_i} \in C_b^{2m-3}(L_t \times W)$, $\forall i = 1,\dots,l$; $W$ an open subset of $\mathbb{R}^l$ such that $u(x) + \theta(v(x) - u(x)) \in W$, $\forall\theta \in [0,1]$. We assume that $2 \leqslant m \leqslant s$; $n + 1 < 2(s-1)$. Then the following inequality is satisfied*

$$\left\| f\big(x, v(x)\big) - f\big(x, u(x)\big)\right\|_{L_t,m-1} \leqslant c_t(m,s) \max_{1\leqslant i\leqslant l} \|D_{u_i} f\|_{C_b^{2m-3}(L_t\times W)}\big[1+\|u\|_{L_t,s}+\|v\|_{L_t,s}\big]^{2m-3}\|v-u\|_{L_t,s-1}, \quad (2)$$

*$c_t(m,s)$ being a non-decreasing function of $t$, depending also on $m$ and $s$.*

(ii) *If in addition to the assumptions in* (i) *we assume that $m \leqslant s-1$, $f \in C_b^{2m-3}(L_t \times W)$, $D_{u_i} f \in C_b^{2m-1}(L_t \times W)$, $\forall i = 1,\dots,l$, and $w \in \mathbb{E}_{s-2}(L_t)$, then it holds that*

$$\begin{aligned}&\left\|\big[f\big(x, v(x)\big) - f\big(x, u(x)\big)\big]w\right\|_{L_t,m-1}\\ &\quad\leqslant c_t(m,s)\max_{1\leqslant i\leqslant l}\|D_{u_i} f\|_{C_b^{2m-1}(L_t\times W)}\big[1+\|u\|_{L_t,s}+\|v\|_{L_t,s}\big]^{2m-1}\|v-u\|_{s-1}^{L_t}\|w\|_{L_t,s-2}.\end{aligned} \quad (3)$$

## 2. The quasilinear Goursat problem: statement and proof of the result

The following quasilinear Cauchy problem is considered with unknown $u$

$$g^{ab}(x,u)D_{ab}u = f(x,u,Du) \quad \text{in } L_T, \qquad u = \underset{\omega}{u} \quad \text{on } G_T^\omega, \quad (4)$$

where $T \in (0, T_0]$, $u = (u^A)$, $Du = (D_a u^A) = (\frac{\partial u^A}{\partial x^a})$, $D_{ab}u = (\frac{\partial^2 u^A}{\partial x^a \partial x^b})$, $f = (f^I)$, $\underset{\omega}{u} = (\underset{\omega}{u}^J)$, $\omega = 1,2$, $a,b,\ldots = 1,\ldots,n+1$, $A, I, J, \ldots = 1,\ldots,N$, Einstein summation convention is understood.

**Assumptions $(q_s)$, $\frac{n}{2}+2 < s \in \mathbb{N}$**

- $g^{ab}(x,u) \in C^{2s-2}(U \times V)$, where $U$ is an open domain of $\mathbb{R}^{n+1}$ and $V$ is an open domain of $\mathbb{R}^N$, such that $L \subset U$, $0 \in V$ and $g^{ab}(x,0) = \gamma^{ab}$ with $\gamma^{ab}$ defined in (6).
- $g^{ab}(x,u)$ is regularly hyperbolic (see [7]) with hyperbolic constant $h$ independent of $(x,u)$,
- $f(x,u,Du) \in C^{2s-4}(U \times V \times W)$, $f(x,0,0) = 0$, where $W$ is an open domain of $\mathbb{R}^{N(n+1)}$,
- $\underset{\omega}{u}$ is continuous on $G^\omega$ and $G^\omega$ is characteristic w.r.t. $g^{ab}(x,\underset{\omega}{u}(x))$, $\underset{\omega}{u}(G_T^\omega) \subset V$, $\underset{\omega}{u} \in E_{2s-1}(G_T^\omega)$, $[\underset{\omega}{u}]_\Gamma \in H_{2s-1}(\Gamma)$, $\underset{1}{u} = \underset{2}{u}$ on $\Gamma$.

**Theorem 2.1.** (i) *Under assumptions $(q_s)$, there exists $T_2 \in (0,T]$ such that the quasilinear Goursat problem* (4) *has in $L_{T_2}$ a unique solution $u \in \mathbb{E}_s(L_{T_2})$.*

(ii) *There exists a positive real number $d$ such that if $\sum_{\omega=1}^2 |\underset{\omega}{u}|_{G_T^\omega,2s-1} < d$, then the solution in* (i) *is global, i.e. $T_2 = T$.*

**Proof.** The proof of item (i) is sketched in four main steps. The details are provided in [9]. Define $\widetilde{\mathbb{E}}_s(L_T) = \{w \in \mathbb{E}_s(L_T)$: $w = \underset{\omega}{u}$ on $G_T^\omega\}$ and consider the following mapping $\kappa : \underset{w\mapsto\kappa(w)=u}{\widetilde{\mathbb{E}}_s(L_T) \to \widetilde{\mathbb{E}}_s(L_T)}$, where $u$ solves the following linear Goursat problem

$$g^{ab}(x,w)D_{ab}u = f(x,w,Dw) \quad \text{in } L_T, \qquad u = \underset{\omega}{u} \quad \text{on } G_T^\omega. \quad (5)$$

At the first step, thanks to Theorem 8.1 of [7], we construct an element $w_1$ of $\widetilde{\mathbb{E}}_s(L_T)$ as the solution of the following linear Goursat problem

$$\gamma^{ab}D_{ab}w = 0 \quad \text{in } L_T, \qquad w = \underset{\omega}{u} \quad \text{on } G_T^\omega, \quad (6)$$

where $\gamma^{ab} = -1$ if $(a,b) = (1,2)$ or $(2,1)$, $\gamma^{ab} = 1$ if $a = b = 3,\ldots,n+1$, $\gamma^{ab} = 0$ elsewhere. At the second step, by using Sobolev inequalities in Section 3 of [7] and Moser estimate (1) of Theorem 1.1, we show that the mapping $\kappa$ is well defined. At the third step we prove that $\kappa$ is a contraction from a ball of $\mathbb{E}_s(L_{T_2})$ into itself. To do this, take $w \in \widetilde{\mathbb{E}}_s(L_T)$, $u = \kappa(w)$, then use once more Sobolev inequalities and Theorem 8.1 of [7] and Moser estimate (1) of Theorem 1.1 to get the following inequality for $t \in (0,T]$:

$$\|u\|_{L_t,s} \leqslant C_1\left[\sum_{\omega=1}^2 |\underset{\omega}{u}|_{G_t^\omega,2s-1} + t^{\frac12}\|f\|_{C^{2s-3}(L_T\times B_{r_1}\times B_{r_2}^*)}\big[1+\|w\|_{L_t,s}\big]^{2s-3}\right], \quad (7)$$

where the constant $C_1 > 0$ does not depend on $t$, $B_{r_1}$ and $B_{r_2}^*$ are such that $B_{r_1} \times B_{r_2}^* \subset V \times W$ where $V$ and $W$ are defined in assumptions $(q_s)$. Taking $R = \max\{\|w_1\|_{L_T,s}, 2C_1\sum_{\omega=1}^2 |\underset{\omega}{u}|_{G_T^\omega,2s-1}\}$, it follows from (7) that there exists $T_1 \leqslant T$ such that

$$\|w\|_{L_t,s} \leqslant R \quad \Rightarrow \quad \|u\|_{L_t,s} \leqslant R \quad \text{for } t \leqslant T_1. \quad (8)$$

Now take $w_1, w_2 \in \widetilde{\mathbb{E}}_s(L_T)$ such that $\|w_i\|_{L_t,s} \leqslant R$ and $u_i = \kappa(w_i)$, $i = 1,2$. By using Sobolev inequalities of [7] and Moser estimates (2) and (3) of Theorem 1.2, we gain the following inequality for $0 < t \leqslant T_1$:

$$\|u_1 - u_2\|_{L_t,s-1} \leqslant Ct^{\frac{1}{2}}(1+R)[1+2R]^{2s-5}\|w_1 - w_2\|_{L_t,s-1}, \tag{9}$$

where the constant $C_1 > 0$ does not depend on $t$. It follows from (9) that there exists a positive real number $T_2 \leqslant T_1$ such that

$$CT_2^{\frac{1}{2}}(1+R)[1+2R]^{2s-5} < \frac{1}{2}. \tag{10}$$

Set $B_{R,T_2} = \{w \in \widetilde{\mathbb{E}}_s(L_{T_2}):\ \|w\|_{L_{T_2},s} \leqslant R\}$. $B_{R,T_2}$ is endowed with the distance defined by the norm $\|\,.\,\|_{L_{T_2},s-1}$ to be a non-empty and complete metric space (thanks to weak compactness arguments as in [5, 7]). It follows from (8), (9) and (10) that $\kappa$ is a contraction from $B_{R,T_2}$ into itself. Thus $\kappa$ has a unique fixed point $u \in B_{R,T_2}$. $u$ is therefore a solution to the Goursat problem (4) in $\mathbb{E}_s(L_{T_2})$. At the fourth step the uniqueness of the solution of (4) in $\mathbb{E}_s(L_{T_2})$ is shown by using the energy inequality for the linear Goursat problem and Gronwall lemma as in [5,7]. The proof of item (ii) follows from item (i) through a judicious exploitation of hypothesis $f(x,0,0) = 0$ and by applying Taylor formula to the function $f(x,.,.)$ in the neighborhood of $(u, Du) = (0,0) \in \mathbb{R}^N \times \mathbb{R}^{(n+1)N}$. $\square$

## 3. Application to the Einstein–Yang–Mills–Higgs system (EYMH)

Here we discuss the local resolution of the Goursat problem for the EYMH system. Throughout all the section Roman indices vary from 1 to 4 whereas Greek indices vary from 3 to 4. Denote by $(x^i)$ the local coordinates in an unknown 4-dimensional manifold $\mathcal{M}$ endowed with an unknown Lorentzian metric $g$. Let $(\varepsilon_I)_{I=1,\dots,N}$ be an orthogonal basis of an $\mathbb{R}$-based Lie algebra $\mathcal{G}$ endowed with an $Ad$-invariant non-degenerate scalar product denoted by a dot ".", which enjoys the following property (see [1]) $f.[k,l] = [f,k].l$, $\forall f,k,l \in \mathcal{G}$. Here [,] denote the Lie brackets of the Lie algebra $\mathcal{G}$. Let $A$ be an unknown 1-form (Yang–Mills potential) defined on $\mathcal{M}$ with values in $\mathcal{G}$. $A$ is locally defined by $A = A_i^I dx^i \otimes \varepsilon_I$, where $A_i^I$ are $\mathbb{R}$-valued functions defined on $\mathcal{M}$. Consider the unknown Yang–Mills field $F$ (the curvature of $A$) which is locally defined by $F_{ij} = \nabla_i A_j - \nabla_j A_i + [A_i, A_j]$, where $\nabla$ denotes the covariant derivative w.r.t. the space–time metric $g$. Consider an unknown Higgs field which is an unknown $\mathcal{G}$-valued function $\Phi$ defined on $\mathcal{M}$. The EYMH system reads as follows

$$\begin{aligned}
&R_{ij} - \frac{1}{2}Rg_{ij} = \rho_{ij} && \text{(Einstein system)},\\
&\widehat{\nabla}_i F^{ik} = J^k && \text{(Yang–Mills system)},\\
&\widehat{\nabla}_i \widehat{\nabla}^i \Phi = H && \text{(Higgs system)},
\end{aligned} \tag{11}$$

where $(R_{ij})$ and $R$ are respectively the Ricci tensor and the scalar curvature of the unknown metric $g$. $(\rho_{ij})$ is the energy–momentum tensor given by $\rho_{ij} = F_{ik}.F_j^k - \frac{1}{4}g_{ij}F_{kl}.F^{kl} + \Phi_{ij}$, where $\Phi_{ij} = \widehat{\nabla}_i\Phi.\widehat{\nabla}_j\Phi - \frac{1}{2}g_{ij}(\widehat{\nabla}_k\Phi.\widehat{\nabla}^k\Phi + V(\Phi^2))$, $\Phi^2 = \Phi.\Phi$, $V$ is a given $C^\infty$ real valued function defined on $\mathbb{R}$. $(J^k)$ is the Yang–Mills current given by $J^k = [\Phi, \widehat{\nabla}^k\Phi]$. $\widehat{\nabla}_i$ is the gauge covariant derivative or the Yang–Mills operator, it acts on $\Phi$ and $F^{ij}$ as follows: $\widehat{\nabla}_i\Phi = \nabla_i\Phi + [A_i, \Phi]$, $\widehat{\nabla}_i F^{ij} = \nabla_i F^{ij} + [A_i, F^{ij}]$. $H$ is a known $C^\infty$ $\mathcal{G}$-valued function given by (see [1]) $H^I = V'(|\Phi|^2)\Phi^I$, where $V'$ is the derivative of $V$. Throughout the remainder of the paper comma "," denotes usual partial derivative (e.g. $\frac{\partial\Phi}{\partial x^i} \equiv \Phi_{,i}$). Assume that the following harmonic and Lorentz gauges conditions are fulfilled

$$\Gamma^k \equiv g^{lm}\Gamma_{lm}^k = 0, \qquad \Delta \equiv \nabla_k A^k = 0, \tag{12}$$

where $\Gamma_{lm}^k$ are the Christoffel symbols of the unknown metric $g$. Then the Einstein–Yang–Mills–Higgs system (11) reduces to the following hyperbolic quasilinear form with unknown $u = (g_{ij}, A_p, \Phi)$:

$$g^{km}D_{km}u = f(u, Du), \tag{13}$$

where the non-linearity $f(u,Du) = (f_{ij}(u,Du), f_p(u,Du), \Psi(u,Du))$ is given as in [9] by $f_{ij}(u,Du) = Q_{ij}(g,Dg) - 2\rho_{ij} + 2g_{ij}(g^{ab}\Phi_{ab})$, where $Q_{ij}$ depends quadratically on $Dg$,

$$\begin{aligned}
f_p(u,Du) &= J_p(A,\Phi,D\Phi) - \big(g^{ki}_{,p}A_{k,i} + g^{ik}[A_k, A_p]_{,i}\big)\\
&\quad + g_{jp}\big[\big(g^{ik}g^{jl}\big)_{,i}\big[A_{l,k} - A_{k,l} + [A_k, A_l]\big] + \Gamma^i_{im}F^{mj} + \Gamma^j_{im}F^{im} + \big[A_i, F^{ij}\big]\big],\\
\Psi(u,Du) &= H(\Phi) - \big(2\big[A_i, \nabla^i\Phi\big] + \big[A_i, \big[A^i, \Phi\big]\big]\big).
\end{aligned}$$

The following theorem summarizes the resolution of the Goursat problem for the EYMH system in weighted spaces defined in Section 1. Both the evolution and the constraints problems are involved. The improvement here is that the data are constructed for the EYMH model and are of finite differentiability order whereas those of [3,8] are $C^\infty$ and constructed either for the vacuum Einstein, Einstein-perfect fluid models or the EYM model.

**Theorem 3.1.** *Let $T \in (0, T_0]$ be a real number, $p \geqslant 4$ an integer and $\omega \in \{1, 2\}$. Let $\underset{\omega 33}{h}, \underset{\omega 34}{h}, \underset{\omega 44}{h} \in K_{2p-1}(G_T^\omega)$ be given functions on $G_T^\omega$ which constitute two symmetric positive definite matrices with determinant 1 at each point and $(\underset{1 33}{h}, \underset{1 34}{h}, \underset{1 44}{h}) = (\underset{2 33}{h}, \underset{2 34}{h}, \underset{2 44}{h})$ on $\Gamma$. Let $\underset{\omega}{\widetilde{\Phi}}, \underset{\omega 3}{\widetilde{A}}, \underset{\omega 4}{\widetilde{A}} \in K_{2p-1}(G_T^\omega)$ such that $(\underset{1}{\widetilde{\Phi}}, \underset{1 3}{\widetilde{A}}, \underset{1 4}{\widetilde{A}}) = (\underset{2}{\widetilde{\Phi}}, \underset{2 3}{\widetilde{A}}, \underset{2 4}{\widetilde{A}})$ on $\Gamma$. Let $\widetilde{\Omega}, \widetilde{\Omega}_1, \widetilde{\Omega}_2, \underset{23}{\widetilde{b}}, \underset{24}{\widetilde{b}}, \underset{2 1}{\widetilde{A}} \in H_{2p-1}(\Gamma)$.*

*Then there exists $T_1 \in (0, T]$; $\underset{\omega}{\Phi}, \underset{\omega}{\Omega}, \underset{\omega ij}{g}, \underset{\omega \alpha}{A} \in E_{2p-1}(G_{T_1}^\omega)$; $\Phi, \Omega, A_k, g_{ij} \in \mathbb{E}_p(L_{T_1})$ such that*

(1) $\underset{\omega \alpha\beta}{g} = \underset{\omega}{\Omega}\, \underset{\omega \alpha\beta}{h}$ *on* $G_{T_1}^\omega$,

(2) $u = (g_{ij}, A_k, \Phi)$ *is the solution of the complete EYMH system* (11) *with initial data* $\underset{\omega}{u} = (\underset{\omega ij}{g}, \underset{\omega k}{A}, \underset{\omega}{\Phi})$ *on* $G_{T_1}^\omega$,

(3) $\underset{1}{\Omega} = \underset{2}{\Omega} = \widetilde{\Omega}$; $\Omega_{,1} = \widetilde{\Omega}_1$; $\Omega_{,2} = \widetilde{\Omega}_2$; $g_{13,2} = \underset{23}{\widetilde{b}}$; $g_{14,2} = \underset{24}{\widetilde{b}}$ *and* $A_{1,2} = \underset{2 1}{\widetilde{A}}$ *on* $\Gamma$.

**Proof.** We begin by discussing the resolution of the evolution problem, i.e. the resolution of the reduced EYMH system (13). Setting $k_{ij} = g_{ij} - \gamma_{ij}$ where $\gamma_{ij}$ is the inverse of $\gamma^{ij}$ defined in (6), one obtains a convenient second order hyperbolic quasilinear system with unknown $(k_{ij}, A_p, \Phi)$ to which Theorem 2.1 applies. After the resolution of (13) we proceed to the resolution of the constraints problem. We just sketch how this problem is solved in three main steps. The details are provided in [9]. Firstly the hierarchical method of A.D. Rendall [8] is used to construct, from appropriate free data, all the remaining $C^\infty$ initial data satisfying the gauge conditions $\Gamma^k = 0$ and $\Delta = 0$ on $G^1 \cup G^2$. Secondly energy inequalities are established in appropriate weighted Sobolev spaces. Thirdly the $C^\infty$ result and energy inequalities are combined to derive, thanks to a contraction argument, the construction of initial data in the weighted Sobolev spaces $E_{2p-1}(G_{T_1}^\omega)$ that appear in the resolution of (13). Finally, since the gauge conditions are fulfilled, it turns out that the solution of (13) is also the solution of the complete EYMH system (11). □

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