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Enumeration of the 50 fake projective planes

Énumération des 50 faux plans projectifs

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ABSTRACT

Building upon the classification of Prasad and Yeung [Invent. Math. 168 (2007) 321–370], we have shown that there exist exactly 50 fake projective planes (up to homeomorphism; 100 up to biholomorphism), and exhibited each of them explicitly as a quotient of the unit ball in $\mathbb{C}^2$. Some of these fake planes admit singular quotients by 3 element groups and three of these quotients are simply connected. Also exhibited are algebraic surfaces with $c_1^2 = 3c_2 = 9n$ for any positive integer n.

RÉSUMÉ

En partant de la classification de Prasad et Yeung [Invent. Math. 168 (2007) 321–370], nous montrons qu'il existe précisément 50 faux plans projectifs (à homéomorphisme près, 100 à biholomorphisme près), et présentons chacun comme un quotient de la boule unité de $\mathbb{C}^2$. Certains de ces plans admettent des quotients singuliers par des groupes d'automorphismes à 3 éléments, et trois d'entre eux sont simplement connexes. De plus, pour chaque entier $n > 0$, nous présentons des surfaces algébriques avec $c_1^2 = 3c_2 = 9n$.

1. Introduction

A *fake projective plane* is a smooth compact complex surface M which is not biholomorphic to the complex projective plane $\mathbb{P}^2_{\mathbb{C}}$, but has the same Betti numbers as $\mathbb{P}^2_{\mathbb{C}}$, namely $1, 0, 1, 0, 1$. Mumford [9] constructed the first such surface and showed that only finitely many exist. Two more examples were found by Ishida and Kato [4], and another by Keum [5]. See Rémy [13] and Yeung [16] for recent surveys.

By [14], the universal cover of a fake projective plane M is the unit ball $B_1(\mathbb{C}^2)$ in $\mathbb{C}^2$. So the fundamental group Π is a cocompact torsion-free discrete subgroup Π of $PU(2,1)$ having finite abelianization. By Mostow's strong rigidity theorem, Π determines M up to holomorphic or anti-holomorphic equivalence. By [7], no fake projective plane can be anti-holomorphic to itself. By the Hirzebruch Proportionality Principle [3], Π must have covolume 1 in $PU(2,1)$. By [8,15], Π must be arithmetic. The algebraic group $\bar{G}(k)$ in which Π is arithmetic is described as follows (see [11]). There is a pair (k, ℓ) of number fields such that k is totally real and ℓ is a totally complex quadratic extension of k. There is a central simple algebra $\mathcal{D}$ of degree 3 with center ℓ and an involution ι of the second kind on $\mathcal{D}$ such that $k = \{x \in \ell\colon\ \iota(x) = x\}$. The algebraic group $\bar{G}$ is defined over k such that $\bar{G}(k) \cong \{z \in \mathcal{D} \mid \iota(z)z = 1\}/\{t \in \ell \mid \bar{t}t = 1\}$. There is one Archimedean place v_0 of k so that

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$\bar{G}(k_{v_0}) \cong PU(2,1)$ and $\bar{G}(k_v)$ is compact for all other Archimedean places v. The data $(k, \ell, \mathcal{D}, v_0)$ determines $\bar{G}$ up to k-isomorphism. Using Prasad's covolume formula [10], Prasad and Yeung [11,12] eliminated most $(k, \ell, \mathcal{D}, v_0)$, and listed a small number of possibilities where Π's might occur.

Moreover, their results (recast slightly) give a short list of maximal arithmetic subgroups $\bar{\Gamma}$ which might contain a Π. Each of these $\bar{\Gamma}$'s has the form $\bar{G}(k) \cap \prod_{v \in V_f} \bar{P}_v$, where V_f denotes the set of non-Archimedean places of k and where $\{\bar{P}_v\colon v \in V_f\}$ is a coherent family of maximal parahoric subgroups $\bar{P}_v \leqslant \bar{G}(k_v)$. For all but three Π's, there is a unique $\bar{\Gamma}$ containing it. In the remaining three cases, Π is contained in two maximal arithmetic subgroups, whose intersection is a group $\bar{G}(k) \cap \prod_{v \in V_f} \bar{P}_v$, where one of the $\bar{P}_v$'s is Iwahori, rather than maximal. In all cases the *class* of Π is specified by k, ℓ, $\mathcal{D}$ and the family $\{\bar{P}_v\colon v \in V_f\}$. For each class there is an integer $N \geqslant 1$ such that the fundamental groups of the fake projective planes are the torsion-free subgroups Π of index N in the corresponding $\bar{G}(k) \cap \prod_{v \in V_f} \bar{P}_v$ having finite abelianization.

One uses lattices to describe concretely the parahoric subgroups $\bar{P}_v$ involved in each class. If $v \in V_f$ splits in ℓ and if $\bar{G}(k_v)$ is not compact, then $\bar{G}(k_v) \cong PGL(3, k_v)$. The maximal parahoric subgroup $\bar{P}_v$ is conjugate to $PGL(3, \mathcal{O}_v)$, where $\mathcal{O}_v$ is the valuation ring in k_v. When $v \in V_f$ does not split in ℓ, denote also by v the unique place of ℓ over v. Let k_v and ℓ_v be the corresponding completions, $\mathcal{O}_v$ the valuation ring in ℓ_v, and π_v a uniformizer of ℓ_v. Then ι induces a nondegenerate hermitian form h_v on ℓ_v^3, and $\bar{G}(k_v) \cong PU(h_v)$. So $\bar{G}(k_v)$ acts on the set of $\mathcal{O}_v$-lattices in ℓ_v^3. The dual $\mathcal{L}'$ of a lattice $\mathcal{L}$ is the lattice $\mathcal{L}' = \{y \in \ell_v^3\colon h_v(x,y) \in \mathcal{O}_v \text{ for all } x \in \mathcal{L}\}$. We shall say that a maximal parahoric $\bar{P}_v$ is *of type 1* if it is the stabilizer of a self-dual lattice $\mathcal{L}_1$, and *of type 2* if it is the stabilizer of a lattice $\mathcal{L}_2$ such that $\pi_v \mathcal{L}_2 \subsetneqq \mathcal{L}_2' \subsetneqq \mathcal{L}_2$. See [2] for further details. A parahoric $\bar{P}_v$ is an Iwahori subgroup if it is the intersection of one maximal parahoric of each type, corresponding to two lattices $\mathcal{L}_1$, $\mathcal{L}_2$ as above, satisfying also $\pi_v \mathcal{L}_2 \subset \mathcal{L}_1 \subset \mathcal{L}_2$.

Let $\mathcal{T}_1$ denote the set of $v \in V_f$ such that v does not split in ℓ and $\bar{P}_v$ is maximal parahoric of type 2. For the 3 classes in which a $\bar{P}_v$ is Iwahori, this happens when v is the 2-adic place; for all other places v' of k not splitting in ℓ, $\bar{P}_{v'}$ is of type 1, and we write $\mathcal{T}_1 = \{2I\}$.

2. Results

We have found a presentation for each relevant $\bar{\Gamma}$, and enumerated the (conjugacy classes of) subgroups Π of index N in $\bar{\Gamma}$ such that Π is torsion-free and has finite abelianization.

When $\mathcal{D}$ splits over ℓ, [11, Proposition 8.8] shows that there are at most 5 possible pairs (k, ℓ), which [11] denotes $\mathcal{C}_1$, $\mathcal{C}_8$, $\mathcal{C}_{11}$, $\mathcal{C}_{18}$ and $\mathcal{C}_{21}$. Our first theorem verifies a conjecture in [11].

Theorem 2.1. *For each of the classes arising from these five field pairs there are no torsion-free subgroups Π of $\bar{\Gamma}$ of index N having finite abelianization. So no fake projective planes occur in these cases.*

In all but one of these classes there is no torsion-free subgroup of $\bar{\Gamma}$ of index N. For the class $(\mathcal{C}_{11}, \mathcal{T}_1 = \emptyset)$, for which $k = \mathbb{Q}(\sqrt{3})$, $\ell = \mathbb{Q}(\sqrt{3}, i)$ and $N = 864$, we show that there is, up to conjugacy, a unique torsion-free subgroup of $\bar{\Gamma}$ of index N. Its abelianization is $\mathbb{Z}^2$. So for each integer $n \geqslant 1$ there is a normal subgroup Π_n of Π of index n. Then [14, Theorem 4] $M_n = B_1(\mathbb{C}^2)/\Pi_n$ satisfies $c_1(M_n)^2 = 3c_2(M_n) = 9n$.

When $\mathcal{D}$ does not split over ℓ, i.e., is a division algebra, it turns out that there is a unique $v \in V_f$ for which $\bar{G}(k_v)$ is compact. This splits over ℓ and lies over the p-adic place of $\mathbb{Q}$, for the p listed in the tables below. Prasad and Yeung [11,12] showed that there are precisely 28 classes, and showed that each is non-empty. The classes are specified by the pairs (k, ℓ) and the p and $\mathcal{T}_1$ listed in Tables 1 and 2.

Theorem 2.2. *Up to automorphisms of $PU(2,1)$, there are precisely 50 subgroups Π of $PU(2,1)$ which are fundamental groups of fake projective planes. The number of Π's in each class is listed in Tables 1 and 2.*

In Tables 1 and 2, $\mathcal{C}_2$, $\mathcal{C}_{10}$, $\mathcal{C}_{18}$ and $\mathcal{C}_{20}$ are notations from [11]. The place 17− of $\mathbb{Q}(\sqrt{2})$ is the 17-adic place for which $\sqrt{2} \equiv -6$. Most of these Π's are congruence subgroups, determined by calculable congruence conditions. However, at least one Π is not a congruence subgroup.

Armed with a presentation of each of the 28 $\bar{\Gamma}$'s, we are able to list not only the subgroups Π of index N, but also the subgroups H such that $\Pi < H \leqslant \bar{\Gamma}$. These give singular surfaces $M_H = B_1(\mathbb{C}^2)/H$ covered by $M = B_1(\mathbb{C}^2)/\Pi$ and having fundamental group $\pi_1(M_H) = H/\langle \text{torsion elements in } H \rangle$ [1]. In particular, the fundamental groups appearing in this way when $[H : \Pi] = 3$ are $\{1\}$, C_2, C_3, C_4, C_6, C_7, C_{13}, C_{14}, $C_2 \times C_2$, $C_2 \times C_4$, S_3, D_8 and Q_8. Here C_n denotes the cyclic group of order n, S_3 is the symmetric group of order 6, and D_8 and Q_8 are the dihedral and quaternionic groups of order 8. In the case $\Pi \lhd H$, Keum [6] obtained much information about the possible M_H from general considerations.

We conclude with a brief description of our methods. In the division algebra case we first realized $\mathcal{D}$ concretely as a cyclic simple algebra over ℓ splitting except at the two places of ℓ corresponding to p. We chose an ι so that $\bar{G}(k_{v_0}) \cong PU(2,1)$ for one Archimedean place v_0 of k (and $\bar{G}(k_v) \cong PU(3)$ at the other Archimedean place v when $[k : \mathbb{Q}] = 2$). For each v we found concrete conditions for an element $g \in \bar{G}(k_v)$ to belong to $\bar{P}_v$ using lattices, as above.

Table 1
The cases $k = \mathbb{Q}$.

ℓ	p	$\mathcal{T}_1$	N	$\sharp\Pi$'s
$\mathbb{Q}(\sqrt{-1})$	5	$\emptyset$	3	1
		$\{2\}$	3	1
		$\{2I\}$	1	1
$\mathbb{Q}(\sqrt{-2})$	3	$\emptyset$	3	1
		$\{2\}$	3	1
		$\{2I\}$	1	1
$\mathbb{Q}(\sqrt{-7})$	2	$\emptyset$	21	3
		$\{7\}$	21	4
		$\{3\}$	3	2
		$\{3,7\}$	3	2
		$\{5\}$	1	1
		$\{5,7\}$	1	1
$\mathbb{Q}(\sqrt{-15})$	2	$\emptyset$	3	2
		$\{3\}$	3	3
		$\{5\}$	3	2
		$\{3,5\}$	3	3
$\mathbb{Q}(\sqrt{-23})$	2	$\emptyset$	1	1
		$\{23\}$	1	1
			Total:	31

Table 2
The cases $k \neq \mathbb{Q}$.

	$k,\ \ell$	p	$\mathcal{T}_1$	N	$\sharp\Pi$'s
$\mathcal{C}_2$	$k = \mathbb{Q}(\sqrt{5})$	2	$\emptyset$	9	6
	$\ell = k(\sqrt{-3})$		$\{3\}$	9	1
$\mathcal{C}_{10}$	$k = \mathbb{Q}(\sqrt{2})$	2	$\emptyset$	3	1
	$\ell = k(\sqrt{-5+2\sqrt{2}})$		$\{17-\}$	3	1
$\mathcal{C}_{18}$	$k = \mathbb{Q}(\sqrt{6})$	3	$\emptyset$	9	1
	$\ell = k(\sqrt{-3})$		$\{2\}$	3	3
			$\{2I\}$	1	1
$\mathcal{C}_{20}$	$k = \mathbb{Q}(\sqrt{7})$	2	$\emptyset$	21	1
	$\ell = k(\sqrt{-1})$		$\{3+\}$	3	2
			$\{3-\}$	3	2
				Total:	19

Computer searches (particularly lengthy in the $\mathcal{C}_{10}$ case) were then done to find sufficiently many elements of $\bar{\Gamma}$ to contain a generating set S. To verify that S generates $\bar{\Gamma}$, we first calculated the radius r_0 and then the volume of the Dirichlet fundamental domain of the subgroup $\langle S\rangle$ generated by S. We checked that this volume matches the covolume of $\bar{\Gamma}$, known from [11], so that $\langle S\rangle = \bar{\Gamma}$. We then enumerated the set of $g \in \bar{\Gamma}$ such that $d(g(0), 0) \leqslant 2r_0$. We used this to (i) find a presentation of $\bar{\Gamma}$ and (ii) list a set of representatives of the conjugacy classes of torsion elements in $\bar{\Gamma}$. We then used Magma (see http://magma.maths.usyd.edu.au/magma/) and GAP (see http://www.gap-system.org) to find all conjugacy classes of subgroups Π of $\bar{\Gamma}$ with the requisite index N. We used (ii) to check which of these were torsion-free. We verified that the abelianization of Π was finite in each case. In the matrix algebra cases, we found finite subgroups K of $\bar{\Gamma}$ and used the fact that if Π is a torsion-free subgroup of $\bar{\Gamma}$ then K acts on $\bar{\Gamma}/\Pi$ without fixed points to exclude the existence of Π of index N and finite abelianization. Many of our results are dependent on computer programs we wrote (see http://www.maths.usyd.edu.au/u/donaldc/fakeprojectiveplanes/).

As an example, let us give some details for the class corresponding to $k = \mathbb{Q}$, $\ell = \mathbb{Q}(\sqrt{-7})$ and $\mathcal{T}_1 = \{7\}$. Let $m = \mathbb{Q}(\zeta)$, where $\zeta = e^{2\pi i/7}$, which is a degree 3 extension of ℓ with Galois group $\mathrm{Gal}(m/\ell) = \langle\varphi\rangle$, where $\varphi(\zeta) = \zeta^2$. Let $\mathcal{D}$ be the central simple algebra over ℓ generated by m and σ, with $\sigma^3 = (3+\sqrt{-7})/4$ and $\sigma x = \varphi(x)\sigma$ for $x \in m$. There is an involution ι_0 of $\mathcal{D}$ of the second kind which maps σ to σ^{-1} and ζ to ζ^{-1}. We replace ι_0 by $\iota : \xi \mapsto w^{-1}\iota_0(\xi)w$, where $w = \zeta + \zeta^{-1}$, to get the desired behaviour $\bar{G}(\mathbb{R}) \cong PU(2,1)$. Then $\bar{\Gamma}$ is generated by ζ and $b = \frac{1}{7}\sum_{j=0}^{5}\sum_{k=-1}^{1} b_{jk}\zeta^j\sigma^k$ for coefficients $-9, -3, 6, -4, 1, -2, 1, -2, -3, -1, -5, 3, -3, -8, 2, 2, -4, -6$ in the order $b_{0,-1}, b_{0,0}, b_{0,1}, b_{1,-1}, \ldots, b_{5,1}$. Mumford's original plane is contained in this class.

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