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Probability Theory

Majorizing measures on metric spaces

Mesures majorantes sur des espaces métriques

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ABSTRACT

In this Note we consider stochastic processes defined on a compact metric space (T,d), with bounded increments in the sense that $\mathbf{E}\varphi(\frac{|X_s-X_t|}{d(s,t)}) \leqslant 1$ for all $s,t \in T$, where φ is an Orlicz function, i.e. is convex, increasing, with $\varphi(0)=0$. We show that whenever d^p is still a metric on T for some $p>1$, then the sample boundedness of all processes with bounded increments can be understood in terms of the existence of a majorizing measure.

RÉSUMÉ

Nous considérons des processus stochastiques définis sur un espace métrique compact (T,d), dont les accroissements sont bornés au sens suivant. On suppose que $\mathbf{E}\varphi(\frac{|X_s-X_t|}{d(s,t)}) \leqslant 1$ pour tous $s,t \in T$, où φ une fonction d'Orlicz, c'est-à-dire convexe, croissante, telle que $\varphi(0)=0$. On suppose que $\mathbf{E}\varphi(\frac{|X_s-X_t|}{d(s,t)}) \leqslant 1$ pour tous $s,t \in T$. Nous montrons que si d^p est encore une distance pour un $p>1$, tous ces processus sont bornés si et seulement s'il existe une certaine mesure majorante sur T.

1. Introduction

Let (T,d) be a compact metric space. We denote the diameter of T by $D(T)$ and an open ball with a center at $t \in T$ and radius r by $B(t,r)$. Let $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ be an Orlicz function, i.e. φ is convex, increasing, $\varphi(0)=0$. We say that the process X_t, $t \in T$, is of bounded increments if for all $s,t \in T$,

$$\mathbf{E}\varphi\left(\frac{|X(s)-X(t)|}{\mathrm{d}(s,t)}\right) \leqslant 1. \tag{1}$$

Note that whenever (1) holds $(X_t)_{t\in T}$ has a separable modification, which we always use when considering the supremum of such a process. The problem, going back to Kolmogorov, is to characterize in terms of geometry of (T,d) whether or not all processes with bounded increments on the space are sample bounded. Such a property (see Talagrand [8]) is equivalent to the following condition:

$$\mathcal{S}(T,d,\varphi) := \sup_X \mathbf{E} \sup_{s,t\in T} \left|X(s)-X(t)\right| < \infty,$$

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where the supremum is taken over all processes that satisfy (1). We recall (see Fernique [3]) that a Borel probability measure m on (T,d) is *majorizing* if

$$\mathcal{M}(m,\varphi)=\sup_{t\in T}\int_0^{D(t)}\varphi^{-1}\left(\frac{1}{m(B(t,r))}\right)\mathrm{d}r<\infty.$$

We also say that m is weakly majorizing if

$$\bar{\mathcal{M}}(m,\varphi)=\int_T\int_0^{D(T)}\varphi^{-1}\left(\frac{1}{m(B(t,r))}\right)\mathrm{d}r<\infty.$$

In [1] (see also [8, Theorem 4.6]) it was proved that the existence of majorizing measure is always sufficient for $S(T,d,\varphi)$ to be finite, namely we have $S(T,d,\varphi)\leqslant 32\mathcal{M}(m,\varphi)$. However it is a non-trivial question to fully characterize Kolmogorov's property. The majorizing measure condition is not necessary, e.g. for the natural distance on subsets of $\mathbb{R}^n$ (see [2,8]). On the other hand the condition is valid in many cases (see [8]). In this paper we follow Fernique's method [4] by which he proved that the majorizing measure condition is necessary for ultrametric spaces. The key tool is the following result:

Theorem 1 *(Fernique). If $\sup_m \bar{\mathcal{M}}(m,\varphi)<\infty$, then there exists a majorizing measure on (T,d). In other words if all measures are weakly majorizing with a uniform constant, then there exists a majorizing measure on (T,d).*

We say that d is regular if there exists a function $\zeta:[0,D(T)]\to\mathbb{R}_+$ such that $\zeta(d)$ is still a metric on T, where ζ is convex, $\zeta(0)=0$ and satisfies the $\triangle_2$-condition, i.e. there exists $C>1$ such that

$$2C\zeta(x)\leqslant\zeta(Cx),\quad \text{for all } 0\leqslant x\leqslant D(T)/C. \tag{2}$$

In particular (2) is satisfied for $\zeta(x)=x^{1/p}$, $p>1$. Our main result is the following:

Theorem 2. *Whenever d is regular and all processes with bounded increments are sample bounded then each Borel m on (T,d) is weakly majorizing and $\sup_m\bar{\mathcal{M}}(m,\varphi)\leqslant 16C\mathcal{S}(T,d,\varphi)$.*

2. Proof of Theorem 2

For a given m we construct (X_t), $t\in T$, with bounded increments that certifies m is weakly majorizing. Note that whenever $\omega\in T$ there exists a point $s\in T$ such that $\mathrm{d}(\omega,s)\geqslant D(T)/2$. We define random variables X_t on the probability space $((T,d),m)$ by

$$X_t(\omega)=c\int_{\mathrm{d}(t,\omega)}^{D(T)/2}\varphi^{-1}\left(\frac{1}{m(B(\omega,2Cr))}\right)\mathrm{d}r,\quad \omega\in T,$$

where we specify $0<c\leqslant 1$ later. Suppose we have proved that (X_t), $t\in T$, verifies (1), then since we have assumed that all processes with bounded increments are sample bounded we learn that

$$\mathbf{E}\sup_{s,t\in T}\left|X(t)-X(s)\right|=c\int_T\int_0^{D(T)/2}\varphi^{-1}\left(\frac{1}{m(B(\omega,2Cr))}\right)\mathrm{d}r\,m(\mathrm{d}\omega)\leqslant\mathcal{S}(T,d,\varphi).$$

Changing variables $r=\varepsilon/(2C)$ we obtain

$$\int_T\int_0^{CD(T)}\varphi^{-1}\left(\frac{1}{m(B(\omega,r))}\right)\mathrm{d}r\,m(\mathrm{d}\omega)\leqslant 2c^{-1}C\mathcal{S}(T,d,\varphi)$$

and therefore $\bar{\mathcal{M}}(\mu,\varphi)\leqslant 2c^{-1}C\mathcal{S}(T,d,\varphi)$. Thus we only need to verify that (1) holds. Since $B(t,Cr)\subset B(\omega,2Cr)$ for $\mathrm{d}(t,\omega)\leqslant r$ and $B(s,Cr)\subset B(\omega,2Cr)$ for $\mathrm{d}(s,\omega)\leqslant r$ we obtain by Jensen's inequality that

$$\mathbf{E}\varphi\left(\frac{|X(s)-X(t)|}{\mathrm{d}(s,t)}\right) \leqslant \int_T \varphi\left(\left|\int_{\mathrm{d}(t,\omega)}^{\mathrm{d}(s,\omega)} \frac{c}{\mathrm{d}(s,t)}\varphi^{-1}\left(\frac{1}{m(B(\omega,2Cr))}\right)\mathrm{d}r\right|\right)m(\mathrm{d}\omega)$$
$$\leqslant \int_T 1_{\{\mathrm{d}(t,\omega)\leqslant \mathrm{d}(s,\omega)\}}\frac{c}{\mathrm{d}(s,t)}\int_{\mathrm{d}(t,\omega)}^{\mathrm{d}(s,\omega)} \frac{1}{m(B(t,Cr))}\,\mathrm{d}r\,m(\mathrm{d}\omega)$$
$$+\int_T 1_{\{\mathrm{d}(s,\omega)\leqslant \mathrm{d}(t,\omega)\}}\frac{c}{\mathrm{d}(s,t)}\int_{\mathrm{d}(t,\omega)}^{\mathrm{d}(s,\omega)} \frac{1}{m(B(s,Cr))}\,\mathrm{d}r\,m(\mathrm{d}\omega). \tag{3}$$

Then by Fubini's theorem $\int_T 1_{\{\mathrm{d}(t,\omega)\leqslant r\leqslant \mathrm{d}(s,\omega)\}}m(\mathrm{d}\omega) = m(B(t,r)\backslash B(s,r))$ and thus

$$\int_T 1_{\{\mathrm{d}(t,\omega)\leqslant \mathrm{d}(s,\omega)\}}\frac{c}{\mathrm{d}(s,t)}\int_{\mathrm{d}(t,\omega)}^{\mathrm{d}(s,\omega)} \frac{1}{m(B(t,Cr))}\,\mathrm{d}r\,m(\mathrm{d}\omega)$$
$$\leqslant \int_0^{D(T)} \frac{c}{\mathrm{d}(s,t)}\frac{m(B(t,r)\backslash B(s,r))}{m(B(t,Cr))}\,\mathrm{d}r. \tag{4}$$

Now we use two different approaches to bound the right-hand side in (4). Clearly

$$\int_T \frac{c}{\mathrm{d}(s,t)}\frac{m(B(t,r)\backslash B(s,r))}{B(t,Cr)}\,\mathrm{d}r \leqslant c. \tag{5}$$

On the other hand the change of variables $r=\zeta^{-1}(\varepsilon)$ implies that

$$\int_{\mathrm{d}(s,t)}^{D(T)} \frac{c}{\mathrm{d}(s,t)}\int_{\mathrm{d}(t\omega)}^{\mathrm{d}(s,\omega)} \frac{1}{m(B(t,Cr))}\,\mathrm{d}r\,m(\mathrm{d}\omega) = \int_{\zeta(\mathrm{d}(s,t))}^{\zeta(D(T))} \frac{cm(B(t,\zeta^{-1}(\varepsilon))\backslash B(s,\zeta^{-1}(\varepsilon)))}{\mathrm{d}(s,t)\zeta'(\zeta^{-1}(\varepsilon))m(B(t,C\zeta^{-1}(\varepsilon)))}\,\mathrm{d}\varepsilon. \tag{6}$$

Using that $\zeta(d)$ is a metric on T we deduce that

$$B\big(t,\zeta^{-1}(\varepsilon)\big)\setminus B\big(s,\zeta^{-1}(\varepsilon)\big) \subset B\big(t,\zeta^{-1}(\varepsilon)\big)\setminus B\big(t,\zeta^{-1}\big(\varepsilon-\zeta\big(\mathrm{d}(s,t)\big)\big)\big),$$

therefore

$$\int_{\zeta(\mathrm{d}(s,t))}^{\zeta(D(T))} \frac{cm(B(t,\zeta^{-1}(\varepsilon))\backslash B(s,\zeta^{-1}(\varepsilon)))}{\mathrm{d}(s,t)\zeta'(\zeta^{-1}(\varepsilon))m(B(t,C\zeta^{-1}(\varepsilon)))}\,\mathrm{d}\varepsilon$$
$$\leqslant \int_{\zeta(\mathrm{d}(s,t))}^{\zeta(D(T))} \frac{m(B(t,\zeta^{-1}(\varepsilon)))-m(B(t,\zeta^{-1}(\varepsilon-\zeta(\mathrm{d}(s,t)))))}{\mathrm{d}(s,t)\zeta'(\zeta^{-1}(\varepsilon))m(B(t,C\zeta^{-1}(\varepsilon)))}\,\mathrm{d}\varepsilon. \tag{7}$$

Let k_0 be such that $C^{-k_0-1}D(T)\leqslant \mathrm{d}(s,t)\leqslant C^{-k_0}D(T)$. Note that for all $C^{-k-1}D(T)<\zeta^{-1}(\varepsilon)\leqslant C^{-k}D(T)$, $0\leqslant k<k_0$, we have

$$\zeta'\big(\zeta^{-1}\big(C^{-k-1}D(T)\big)\big)m\big(B\big(t,C^{-k}D(T)\big)\big) \leqslant \zeta'\big(\zeta^{-1}(\varepsilon)\big)m\big(B\big(t,C\zeta^{-1}(\varepsilon)\big)\big) \tag{8}$$

and

$$\int_{\zeta(C^{-k-1}D(T))}^{\zeta(C^{-k}D(T))} \big(m\big(B\big(t,\zeta^{-1}(\varepsilon)\big)\big)-m\big(B\big(t,\zeta^{-1}\big(\varepsilon-\zeta\big(\mathrm{d}(s,t)\big)\big)\big)\big)\big)\,\mathrm{d}\varepsilon \leqslant \zeta\big(\mathrm{d}(s,t)\big)m\big(B\big(t,C^{-k}D(T)\big)\big). \tag{9}$$

Combining (8) and (9) we obtain that

$$\int_{\zeta(C^{-k-1}D(T))}^{\zeta(C^{-k}D(T))} \frac{m(B(t,\zeta^{-1}(\varepsilon)))-m(B(t,\zeta^{-1}(\varepsilon-\zeta(\mathrm{d}(s,t)))))}{\mathrm{d}(s,t)\zeta'(\zeta^{-1}(\varepsilon))m(B(t,C\zeta^{-1}(\varepsilon)))}\,\mathrm{d}\varepsilon \leqslant \frac{c\zeta(\mathrm{d}(s,t))}{\mathrm{d}(s,t)\zeta'(C^{-k-1}D(T))} \tag{10}$$

for all $0 \leqslant k < k_0$. Similarly we show that

$$\int_{\zeta(\mathrm{d}(s,t))}^{\zeta(C^{-k_0}D(T))} \frac{m(B(t,\zeta^{-1}(\varepsilon))) - m(B(t,\zeta^{-1}(\varepsilon - \zeta(\mathrm{d}(s,t)))))}{\mathrm{d}(s,t)\zeta'(\zeta^{-1}(\varepsilon))m(B(t,C\zeta^{-1}(\varepsilon)))}\,\mathrm{d}\varepsilon \leqslant \frac{c\zeta(\mathrm{d}(s,t))}{\mathrm{d}(s,t)\zeta'(\mathrm{d}(s,t))}. \tag{11}$$

Using that C is the constant in (2) we obtain that

$$\frac{c\zeta(\mathrm{d}(s,t))}{\mathrm{d}(s,t)\zeta'(C^{-k-1}D(T))} \leqslant \frac{c\zeta(\mathrm{d}(s,t))C^{-k-1}D(T)}{\mathrm{d}(s,t)\zeta(C^{k-1}D(T))} \leqslant \frac{c}{2^{k_0-1-k}} \tag{12}$$

for all $0 \leqslant k < k_0$. Inequalities (6), (7), (10), (11) and (12) lead to

$$\int_{\mathrm{d}(s,t)}^{D(T)} \frac{c}{\mathrm{d}(s,t)} \frac{m(B(t,r)\backslash B(s,r))}{m(B(t,Cr))}\,\mathrm{d}r \leqslant c + \sum_{k=0}^{k_0-1} \frac{c}{2^{k_0-1-k}} \leqslant 3c. \tag{13}$$

Plugging (5) and (13) into (4) and using (3) we obtain that

$$\mathbf{E}\varphi\left(\frac{|X_s - X_t|}{\mathrm{d}(s,t)}\right) \leqslant 8c, \quad \text{for all } s,t \in T.$$

Taking $c = 1/8$ completes the proof.

3. Applications

The main application of the result is to the case $T \subset \mathbb{R}^n$ with the metric $\mathrm{d}(s,t) = \|s-t\|^{1/p}$, for $p > 1$. Note that Theorem 2 characterizes the sample boundedness of all processes $(X_t)_{t\in T}$ such that $\mathbf{E}|X_s - X_t|^p \leqslant \|s-t\|$ in terms of the existence of a majorizing measure.

Corollary 1. *For $T \subset \mathbb{R}^n$ with $\mathrm{d}(s,t) = \|s-t\|^{1/p}$, $p > 1$, all processes that satisfy* (1) *are sample bounded if and only if there exists a majorizing measure on (T,d) (for $\varphi(x) \equiv x^p$).*

This generalizes previous results in this direction (see [8, Section 5]). A closely related question (see [5]) is the characterization of coefficients $(a_n)_{n\geqslant 1}$ such that $\sum_{n=1}^{\infty} a_n^2 = 1$ and $\sum_{n=1}^{\infty} a_n\varphi_n$ is a.s. convergent for each orthonormal $(\varphi_n)_{n\geqslant 1}$. Defining $T_a = \{\sum_{n=m}^{\infty} a_n^2,\ m \geqslant 1\}$, with $\mathrm{d}(s,t) = \sqrt{|s-t|}$, we note that processes $X_t = \sum_{n\geqslant m} a_n\varphi_n$ for $t = t_m = \sum_{n\geqslant m} a_n^2$ satisfy $\mathbf{E}|X_s - X_t|^2 = \mathrm{d}^2(s,t)$ for $s,t \in T_a$. On the other hand each process such that $\mathbf{E}|X_s - X_t|^2 = \mathrm{d}^2(s,t) = |s-t|$ can be represented as $X_t = \sum_{n\geqslant m} a_n Y_n$, where $Y_m = (X_{t_m} - X_{t_{m+1}})/a_m$ are clearly orthonormal. Therefore the problem can be reformulated in terms of the sample boundedness of all processes on T_a such that $\mathbf{E}|X_s - X_t|^2 = \mathrm{d}^2(s,t) = |s-t|$. The question has a long history and partial results were given in [9–12]. A complete solution is obtained by Paszkiewicz in [7] (see also [6]). Our result – Corollary 1 – implies that the majorizing measure condition is necessary for the bigger class of processes to be sample bounded namely for all $(X_t)_{t\in T_a}$ such that $\mathbf{E}|X_s - X_t|^2 \leqslant |s-t|$.

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