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Group Theory/Topology

Finiteness properties for a subgroup of the pure symmetric automorphism group

Propriétés de finitude pour un sous-groupe du groupe des automorphismes symétriques et purs

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ABSTRACT

Let F_n be the free group on n generators, and let $P\Sigma_n$ be the group of automorphisms of F_n that send each generator to a conjugate of itself. The kernel K_n of the homomorphism $P\Sigma_n \to P\Sigma_{n-1}$, induced by mapping one of the free group generators to the identity, is finitely generated. We show that K_n has cohomological dimension $n-1$, and that $H_i(K_n;\mathbb{Z})$ is not finitely generated for $2 \leqslant i \leqslant n-1$. It follows that K_n is not finitely presentable for $n \geqslant 3$.

RÉSUMÉ

Soit F_n le groupe libre engendré par n éléments, et soit $P\Sigma_n$ le groupe des automorphismes de F_n qui envoient chaque générateur sur un conjugué. Le noyau K_n de l'homomorphisme $P\Sigma_n \to P\Sigma_{n-1}$, obtenu en envoyant un des générateurs du groupe libre sur l'identité, est de type fini. On démontre que K_n est de dimension cohomologique $n-1$, est que $H_i(K_n;\mathbb{Z})$ n'est pas de type fini pour $2 \leqslant i \leqslant n-1$. Par conséquent K_n n'est pas de présentation finie pour $n \geqslant 3$.

1. Introduction

In a recent work, Brendle and Hatcher [2] proved that the space of all smooth links in $\mathbb{R}^3$ isotopic to the trivial link of n components has the homotopy type of the finite-dimensional subspace of configurations of n unlinked circles, and thus their fundamental groups are isomorphic. The fundamental group of the latter space is a 3-dimensional analogue of the classical braid group (the space of configurations of n points in $\mathbb{R}^2$), and Goldsmith [6] showed that it is isomorphic to the *symmetric automorphism group*, the group of automorphisms of F_n which send every generator to a conjugate of another generator or its inverse.

The subgroup consisting of those automorphisms which send every generator to a conjugate of itself (or, in mapping class group terms, those classes which send every oriented circle in $\mathbb{R}^3$ back to itself) is known as the *pure symmetric automorphism group*, denoted by $P\Sigma_n$. McCool [9] gave a finite presentation for $P\Sigma_n$, and Brownstein and Lee [3] computed its cohomology when $n = 3$. Collins [4] proved that $P\Sigma_n$ has cohomological dimension $n-1$; it also follows from his

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work that $\mathrm{P}\Sigma_n$ is FP_∞. Later, Brady, McCammond, Meier and Miller [1] showed that $\mathrm{P}\Sigma_n$ is a duality group, and Jensen, McCammond and Meier [7] determined completely the structure of the cohomology ring of $\mathrm{P}\Sigma_n$ for $n \geqslant 3$.

Let PB_n denote the pure braid group, the elements of the braid group that send each puncture back to itself. It is well known that for all n there is a homomorphism $\pi : PB_n \to PB_{n-1}$ induced by "filling in" a puncture. In fact, there is the following split exact sequence:

$$1 \longrightarrow F_{n-1} \longrightarrow PB_n \xrightarrow{\pi} PB_{n-1} \longrightarrow 1 \tag{1}$$

In particular, the pure braid group may be regarded as an iteration of semi-direct products of free groups. The pure braid group PB_n is isomorphic to a subgroup of $\mathrm{P}\Sigma_n$, and by "filling in" the nth circle we obtain a split exact sequence compatible with (1):

$$1 \longrightarrow K_n \longrightarrow \mathrm{P}\Sigma_n \xrightarrow{\pi} \mathrm{P}\Sigma_{n-1} \longrightarrow 1 \tag{2}$$

For $n = 2$ the kernel K_2 is equal to $\mathrm{P}\Sigma_2$. For $n \geqslant 2$, the group K_n is finitely generated (compare with Lemma 2.1 below), and hence $H_1(K_n;\mathbb{Z})$ is finitely generated. The main purpose of this Note is to study the higher homology groups of K_n for $n \geqslant 3$:

Theorem 1.1. *The group K_n has cohomological dimension $n-1$. For $n \geqslant 3$ its i-th homology group $H_i(K_n;\mathbb{Z})$ is not finitely generated for $2 \leqslant i \leqslant n-1$.*

Collins and Gilbert proved that K_3 is not finitely presentable in [5]. Theorem 1.1 yields an independent proof of this fact, generalizing to all $n \geqslant 3$:

Corollary 1.1. *K_n is not finitely presentable for $n \geqslant 3$.*

As pointed out by Brendle and Hatcher [2], the corollary suggests that these kernels K_n are unlikely to have nice interpretations in terms of configuration spaces of circles.

2. Finitely generated homology groups

In this section we verify the finite generation of K_n and compute its first homology group, $H_1(K_n;\mathbb{Z})$, and its cohomological dimension.

Lemma 2.1. *The group K_n is finitely generated, and its first homology group $H_1(K_n;\mathbb{Z}) \simeq \mathbb{Z}^{2n-2}$.*

Proof. McCool [9] proved that the group $\mathrm{P}\Sigma_n$ is generated by

$$\alpha_{ij}(x_r) = \begin{cases} x_r, & r \neq i \\ x_j x_i x_j^{-1}, & r = i \end{cases} \quad \text{with relators} \quad [\alpha_{ij}, \alpha_{kl}], \quad [\alpha_{ik}, \alpha_{jk}], \quad [\alpha_{ij}, \alpha_{ik}\alpha_{jk}]$$

for distinct i, j, k, and l. It is clear that K_n is normally generated by $\{\alpha_{in}, \alpha_{ni} \mid 1 \leqslant i \leqslant n-1\}$. In fact by examining the McCool relators, these elements are seen to generate K_n:

$$\begin{aligned}
&\alpha_{ij}^{\pm 1}\alpha_{ni}\alpha_{ij}^{\mp 1} = \alpha_{nj}^{\mp 1}\alpha_{ni}\alpha_{nj}^{\pm 1}, \qquad \alpha_{jk}^{\pm 1}\alpha_{ni}\alpha_{jk}^{\mp 1} = \alpha_{ni}, \qquad \alpha_{ji}^{\pm 1}\alpha_{ni}\alpha_{ji}^{\mp 1} = \alpha_{ni} \\
&\alpha_{ij}^{\pm 1}\alpha_{in}\alpha_{ij}^{\mp 1} = \alpha_{nj}^{\mp 1}\alpha_{in}\alpha_{nj}^{\pm 1}, \qquad \alpha_{jk}^{\pm 1}\alpha_{in}\alpha_{jk}^{\mp 1} = \alpha_{in} \\
&\alpha_{ji}^{-1}\alpha_{in}\alpha_{ji} = \alpha_{ji}^{-1}\alpha_{jn}^{-1}\alpha_{ji}\alpha_{in}\alpha_{jn} = \alpha_{ni}\alpha_{jn}^{-1}\alpha_{ni}^{-1}\alpha_{in}\alpha_{jn} \\
&\alpha_{ji}\alpha_{in}\alpha_{ji}^{-1} = \alpha_{jn}\alpha_{in}\alpha_{ji}\alpha_{jn}^{-1}\alpha_{ji}^{-1} = \alpha_{jn}\alpha_{in}\alpha_{ni}^{-1}\alpha_{jn}^{-1}\alpha_{ni}
\end{aligned}$$

The last two expressions are each derived from a conjugate of a McCool relator, followed by a substitution using a second relator.

Consider the free group $F(\{\alpha_{in}, \alpha_{ni}\})$ of rank $2n-2$ on the generators of K_n, a subgroup of the free group $F(\{\alpha_{ij}\})$ of rank $n^2 - n$ on the generators of $\mathrm{P}\Sigma_n$. It is clear from McCool's presentation that the kernel of the map $F(\{\alpha_{ij}\}) \to \mathrm{P}\Sigma_n$ is contained in the commutator subgroup. An element in the kernel of $F(\{\alpha_{in}, \alpha_{ni}\}) \to K_n$ lies in

$$\big[F\big(\{\alpha_{ij}\}\big), F\big(\{\alpha_{ij}\}\big)\big] \cap F\big(\{\alpha_{in}, \alpha_{ni}\}\big)$$

and such an element must also lie in the commutator subgroup of $F(\{\alpha_{in}, \alpha_{ni}\})$. This shows that $H_1(K_n;\mathbb{Z}) \simeq H_1(F(\{\alpha_{in}, \alpha_{ni}\});\mathbb{Z})$, completing the proof of Lemma 2.1. □

Jensen and Wahl [8] describe an $(n-1)$-dimensional contractible simplicial complex X_n on which $P\Sigma_n$ acts freely with compact quotient. Briefly, this complex X_n is the geometric realization of the poset of symmetric based graphs with fundamental group F_n, and a marking from a basis $\{x_1,\dots,x_n\}$ to each graph Γ which induces an isomorphism $F_n \to \pi_1(\Gamma)$. A symmetric graph is one in which every edge belongs to a unique cycle, and the partial ordering is given by the collapsing of edges. The complex X_n embeds into the spine of Autre space, the based-graph version of Culler–Vogtmann's Outer space. We refer the reader to [8] for details.

Lemma 2.2. *K_n has cohomological dimension $n-1$.*

Proof. The existence of X_n gives an upper bound of $n-1$ for the cohomological dimension of K_n. The elements $\{\alpha_{1n},\dots,\alpha_{n-1,n}\}$ generate a free abelian subgroup of rank $n-1$, so that $n-1$ is also a lower bound. This completes the proof of the lemma, and thereby the first part of Theorem 1.1. □

3. Non-finitely generated homology groups

We begin with a short lemma about $H_{n-1}(K_n;\mathbb{Z})$:

Lemma 3.1. *The group $H_{n-1}(K_n;\mathbb{Z})$ has a nontrivial element.*

Proof. The subgroup K_n contains the $n-1$ commuting elements $\alpha_{1n},\dots,\alpha_{n-1,n}$. From the McCool relations, we can verify that we have homomorphisms

$$\mathbb{Z}^{n-1} \longrightarrow K_n \longrightarrow \mathbb{Z}^{n-1}$$

whose composition is the identity. Therefore the induced map $H_{n-1}(\mathbb{Z}^{n-1};\mathbb{Z}) \to H_{n-1}(K_n;\mathbb{Z})$ is injective. □

We next prove a proposition which, together with Lemma 3.1, proves the theorem. The author is thankful to A. Hatcher for suggesting this proposition as a very nice simplification of arguments in an earlier version of this Note:

Proposition 3.1. *Let Γ be a group acting freely and simplicially on a contractible $(n-1)$-dimensional complex X, and let K be normal subgroup of Γ of infinite index. Then if $H_{n-1}(K;\mathbb{Z})$ is nonzero, it is not finitely generated.*

Proof. By assumption, K acts freely on the contractible complex X, so $Y = X/K$ is an Eilenberg–MacLane space of type $K(K,1)$. Thus by the assumption that $H_{n-1}(K;\mathbb{Z}) \neq 0$, we have $H_{n-1}(Y;\mathbb{Z}) \neq 0$. A nontrivial $(n-1)$-cycle of Y is represented by a finite sum of $(n-1)$-simplices, so there exists a nontrivial finite subcomplex A of Y such that $H_{n-1}(A;\mathbb{Z}) \neq 0$. As Γ/K acts freely on Y, and as K has infinite index in Γ, there is an infinite set of pairwise disjoint translates of A by Γ/K; denote the union of such a set of translates by U. Clearly $H_{n-1}(U;\mathbb{Z})$ is not finitely generated. The proof is complete by the following exact sequence on the relative pair (Y,U):

$$\cdots \longrightarrow H_n(Y,U;\mathbb{Z}) \longrightarrow H_{n-1}(U;\mathbb{Z}) \longrightarrow H_{n-1}(Y;\mathbb{Z}) \longrightarrow H_{n-1}(Y,U;\mathbb{Z}) \longrightarrow \cdots$$

The first term $H_n(Y,U;\mathbb{Z}) = 0$ as Y has dimension $n-1$. □

For $n \geqslant 3$, the subgroup K_n has infinite index in $P\Sigma_n$, and so Lemma 3.1 and Proposition 3.1 applied to the Jensen–Wahl complex X_n imply that $H_{n-1}(K_n;\mathbb{Z})$ is not finitely generated. Now observe that there is a split surjective homomorphism $K_n \to K_{n-1}$. This induces a split surjection $H_i(K_n;\mathbb{Z}) \to H_i(K_{n-1};\mathbb{Z})$ for all i. Theorem 1.1 then holds for $n \geqslant 3$ by induction on n.

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