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Partial Differential Equations/Functional Analysis

Almost sure convergence of some random series

Convergence presque sûre de certaines séries aléatoires

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ABSTRACT

Let $(c_n)_{n\geqslant 1}$ be a square-summable sequence of complex numbers, $d \geqslant 2$ an integer, and $(e_{n,d})_{n\geqslant 1}$ the orthonormal basis of the space $L^2([0,1], r^{d-1}\,dr)$ consisting of the radial eigenfunctions of the Laplace operator acting on the space $L^2(D^d)$ of square-summable functions on the unit ball $D^d = \{x \in \mathbb{R}^d;\ r = |x| < 1\}$ of $\mathbb{R}^d$. We generalize a result of Ayache and Tzvetkov and compute in the general case the *critical exponent* of the sequence $(c_n)_{n\geqslant 1}$, i.e. the infimum of the p's, $p \geqslant 2$, such that the random series $\sum \varepsilon_n(\omega)c_n e_{n,d}$ converges almost surely in $L^p([0,1], r^{d-1}c\,dr)$, where (ε_n) denotes a sequence of independent random choices of signs on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

RÉSUMÉ

Soit $(c_n)_{n\geqslant 1}$ une suite de nombres complexes de carré sommable, $d \geqslant 2$ un entier, et $(e_{n,d})_{n\geqslant 1}$ la base orthonormée de l'espace $L^2([0,1], r^{d-1}\,dr)$ formée par les fonctions propres radiales de l'opérateur de Laplace agissant sur l'espace $L^2(D^d)$ des fonctions de carré intégrable sur la boule unité $D^d = \{x \in \mathbb{R}^d;\ r = |x| < 1\}$ de $\mathbb{R}^d$. Nous généralisons un résultat d'Ayache et Tzvetkov en calculant dans le cas général l'*exposant critique* de la suite $(c_n)_{n\geqslant 1}$, c'est-à-dire l'infimum des $p \geqslant 2$ tels que la série aléatoire $\sum \varepsilon_n(\omega)c_n e_{n,d}$ converge presque sûrement dans $L^p([0,1], r^{d-1}\,dr)$, où (ε_n) désigne une suite de choix de signes indépendants sur un espace de probabilité $(\Omega, \mathcal{F}, \mathbb{P})$.

Version française abrégée

Soit $(Y, \mathcal{M}, \mu)$ un espace mesuré de mesure finie, et $(e_n)_{n\geqslant 1}$ une base orthonormée de l'espace $L^2(Y, \mathcal{M}, \mu)$ ayant la propriété suivante : pour tout $p \geqslant 2$ et tout $n \geqslant 1$, e_n appartient à $L^p(Y, \mathcal{M}, \mu)$. Nous étudions dans cette Note les propriétés de convergence de certaines séries aléatoires de la forme $\sum_{n\geqslant 1} \varepsilon_n(\omega)c_n e_n$, où $(c_n)_{n\geqslant 1}$ est une suite de nombres complexes de carré sommable et $(\varepsilon_n)_{n\geqslant 1}$ est une suite de variables aléatoires de Rademacher (c'est-à-dire de choix de signes) indépendantes sur un espace de probabilité $(\Omega, \mathcal{F}, \mathbb{P})$. Le cas le plus étudié d'une telle situation est celui où Y est le tore de dimension d, $\mathbb{T}^d = \mathbb{R}^d/(2\pi\mathbb{Z})^d$, muni de la mesure de Haar, et $(e_n)_{n\in\mathbb{Z}^d}$ est le système trigonométrique dans $L^2(\mathbb{T}^d)$: pour tout multi-indice $n = (n_1, \ldots, n_d)$, $e_n(x_1, \ldots, x_d) = e^{i(n_1x_1+\cdots+n_dx_d)}$. Dans ce cas la série $\sum_{n\in\mathbb{Z}^d} \varepsilon_n(\omega)c_n e_n$ converge presque sûrement dans $L^p(\mathbb{T}^d)$ pour tout $p \geqslant 2$. Mais la situation peut être complètement différente lorsque l'on considère d'autres bases orthonormées. Il est tout à fait possible que la série $\sum_{n\geqslant 1} \varepsilon_n(\omega)c_n e_n$ diverge presque sûrement

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dans $L^p(Y,\mathcal{M},\mu)$ pour un certain $p>2$. Un exemple d'une telle situation est présenté par Ayache et Tzvetkov dans l'article [1], où la définition suivante est introduite :

Définition 0.1. Soit $c=(c_n)_{n\geqslant 1}$ une suite de $\ell^2(\mathbb{N})$. L'*exposant critique* de la suite c relativement à la base orthonormée $(e_n)_{n\geqslant 1}$ est défini comme

$$p_{cr}(c)=\sup\left\{p\geqslant 2;\ \text{la série} \sum_{n\geqslant 1}\varepsilon_n(\omega)c_ne_n \text{ converge presque sûrement dans } L^p(Y,\mathcal{M},\mu)\right\}.$$

Cet exposant critique est calculé dans [1] pour certaines suites c de carré sommable dans le cas où pour un certain entier $d\geqslant 2$, $(e_{n,d})_{n\geqslant 1}$ est la base orthonormée formée par les fonctions propres radiales de l'opérateur de Laplace agissant sur $L^2(D^d)$. Nous calculons l'exposant critique $p_{cr}(c)$ pour toute suite $c\in\ell^2(\mathbb{N})$:

Théorème 0.2. *Soit $c=(c_n)_{n\geqslant 1}$ une suite non identiquement nulle arbitraire de $\ell^2(\mathbb{N})$, et soit*

$$\alpha_*(c)=\inf\left\{\alpha>0;\ \limsup_{N\to+\infty}\frac{1}{N^\alpha}\sum_{n=1}^{N}n^{d-1}|c_n|^2<\infty\right\}.$$

Alors l'exposant critique de c relativement à la base orthonormée $(e_{n,d})_{n\geqslant 1}$ est

$$p_{cr}(c)=\frac{2d}{\alpha_*(c)}.$$

Si au lieu de considérer des séries aléatoires de Rademacher $\sum_{n\geqslant 1}\varepsilon_n(\omega)c_ne_n$ on considère des séries aléatoires gaussiennes $\sum_{n\geqslant 1}g_n(\omega)c_ne_n$, où $(g_n)_{n\geqslant 1}$ est une suite de variables aléatoires gaussiennes standard complexes indépendantes sur $(\Omega,\mathcal{F},\mathbb{P})$, le résultat reste le même : ceci s'ensuit du fait que $L^p(Y,\mathcal{M},\mu)$ est de cotype fini dans le cas $(p>2)$ que nous considérons, et d'un résultat de Maurey et Pisier [4].

1. Introduction

Let $(Y,\mathcal{M},\mu)$ be a finite measure space. Suppose that $(e_n)_{n\geqslant 1}$ is an orthonormal basis of the space $L^2(Y,\mathcal{M},\mu)$ which has the property that for every $p\geqslant 2$, e_n belongs to $L^p(Y,\mathcal{M},\mu)$ for every $n\geqslant 1$. Our aim in this Note is to investigate the convergence properties of random series of the form

$$\sum_{n\geqslant 1}\varepsilon_n(\omega)c_ne_n$$

where $(c_n)_{n\geqslant 1}$ is a sequence of $\ell^2(\mathbb{N})$ and $(\varepsilon_n)_{n\geqslant 1}$ is a sequence of independent Rademacher variables on some probability space $(\Omega,\mathcal{F},\mathbb{P})$, i.e. random choices of signs: $\mathbb{P}(\varepsilon_n=1)=\mathbb{P}(\varepsilon_n=-1)=\frac{1}{2}$. A very well-known instance of this situation is the case where Y is the d-dimensional torus $\mathbb{T}^d=\mathbb{R}^d/(2\pi\mathbb{Z})^d$ endowed with the Haar measure, and $(e_n)_{n\in\mathbb{Z}^d}$ is the trigonometric system in $L^2(\mathbb{T}^d)$: for $n=(n_1,\dots,n_d)$, $e_n(x_1,\dots,x_d)=e^{i(n_1x_1+\cdots+n_dx_d)}$. Then the series $\sum_{n\in\mathbb{Z}^d}g_n(\omega)c_ne_n$ converges almost surely in $L^p(\mathbb{T}^d)$ for every $p\geqslant 2$. But the situation can be quite different when other orthonormal bases are considered. In particular, it is possible that the series $\sum_{n\geqslant 1}\varepsilon_n(\omega)c_ne_n$ diverges almost surely in $L^p(Y,\mathcal{M},\mu)$ for some (or even every) $p>2$. An example of this situation is studied in [1], where the following definition is introduced:

Definition 1.1. Let $c=(c_n)_{n\geqslant 1}$ be an element of $\ell^2(\mathbb{N})$. The *critical exponent* of the sequence c with respect to the orthonormal basis $(e_n)_{n\geqslant 1}$ is defined as

$$p_{cr}(c)=\sup\left\{p\geqslant 2;\ \text{the series} \sum_{n\geqslant 1}\varepsilon_n(\omega)c_ne_n \text{ converges almost surely in } L^p(Y,\mathcal{M},\mu)\right\}.$$

In a Hilbert space H, a random series $\sum_{n\geqslant 1}\varepsilon_n(\omega)x_n$, $x_n\in H$, converges a.s. if and only if the series $\sum_{n\geqslant 1}\|x_n\|^2$ is convergent, and thus the series above converges a.s. in $L^2(Y,\mathcal{M},\mu)$ for every $c\in\ell^2(\mathbb{N})$. The motivation of this Note comes from the work [1] of Ayache and Tzvetkov. Here the critical exponent of some ℓ^2-sequences is studied in the case where for some integer $d\geqslant 2$, $(e_{n,d})_{n\geqslant 1}$ is the orthonormal basis consisting of the radial eigenfunctions of the Laplace operator acting on $L^2(D^d)$: for $r\in(0,1)$,

$$e_{n,d}(r)=\beta_{n,d}^{-1}r^{-\frac{d-2}{2}}J_{\frac{d-2}{2}}(z_{n,d}r),$$

where $J_{\frac{d-2}{2}}$ is the Bessel function of order $\frac{d-2}{2}$, $(z_{n,d})_{n\geqslant 1}$ the increasing sequence of its zeroes on $(0,+\infty)$, and $\beta_{n,d}$ is the normalization constant. The critical exponent of c with respect to $(e_{n,d})_{n\geqslant 1}$ is computed in [1] for sequences c such that there exist two positive constants γ_1 and γ_2 and an $\alpha_0 > 1/2$ such that $\frac{\gamma_1}{n^{\alpha_0}} \leqslant |c_n| \leqslant \frac{\gamma_2}{n^{\alpha_0}}$ for every $n \geqslant 1$. Our main result is the computation of the critical exponent $p_{cr}(c)$ for every sequence $c \in \ell^2(\mathbb{N})$:

Theorem 1.2. *Let $c = (c_n)_{n\geqslant 1}$ be an arbitrary non-zero sequence of $\ell^2(\mathbb{N})$ and define*

$$\alpha_*(c) = \inf\left\{\alpha > 0;\ \limsup_{N\to+\infty} \frac{1}{N^\alpha} \sum_{n=1}^{N} n^{d-1}|c_n|^2 < \infty\right\}. \tag{1}$$

Then the critical exponent of c with respect to the orthonormal basis $(e_{n,d})_{n\geqslant 1}$ is

$$p_{cr}(c) = \frac{2d}{\alpha_*(c)}.$$

Observe that in any case $\alpha_*(c) \leqslant d-1$, and thus $p_{cr}(c) \geqslant 2d/(d-1)$. When for some $\alpha_0 > 1/2$ the sequence $(n^{\alpha_0}|c_n|)$ is bounded and bounded away from zero, this gives again $p_{cr}(c) = 2d/(d-2\alpha_0)$ [1].

We recall in Section 2 some general properties of Banach-valued random sums which put into perspective the approach of [1], and prove Theorem 1.2 in Section 3.

2. General properties of random sums

Let $\sum_{n\geqslant 1} \chi_n(\omega)x_n$ be a random series on a (complex) Banach space X, where the vectors x_n belong to X and the χ_n's are independent and identically distributed (complex-valued) symmetric random variables on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Two particular choices for the random sequence (χ_n) are especially interesting: when $(\chi_n) = (\varepsilon_n)$ is a sequence of independent Rademacher variables on Ω, and when $(\chi_n) = (g_n)$ is a sequence of independent standard (complex-valued) gaussian variables: for every (measurable) subset A of $\mathbb{C}$,

$$\mathbb{P}(g_n \in A) = \frac{1}{2\pi} \int_A e^{-\frac{|z|^2}{2}}\, \mathrm{d}A(z),$$

where $\mathrm{d}A(z)$ is the area measure on $\mathbb{C}$. Alternatively, $g_n = (g_n' + i g_n'')/\sqrt{2}$, where g_n' and g_n'' are independent standard real-valued gaussian variables.

The Kahane inequalities, which generalize the classical Khinchine inequalities, are fundamental for the study of the convergence in $L^p(\Omega; X)$, $1 \leqslant p < +\infty$, of random X-valued Rademacher series, and they imply that the series $\sum_{n\geqslant 1} \varepsilon_n(\omega)x_n$ converges a.s. in X if and only if it converges in some/every $L^p(\Omega; X)$, $p \geqslant 1$. This easily gives a characterization of the almost sure convergence of random series of functions of $L^p(Y, \mathcal{M}, \mu)$, $p \geqslant 1$:

Fact 2.1. *Let $p \geqslant 1$, and let $(f_n)_{n\geqslant 1}$ be a sequence of elements of $L^p(Y, \mathcal{M}, \mu)$. The series $\sum_{n\geqslant 1} \varepsilon_n(\omega) f_n$ converges a.s. in $L^p(Y, \mathcal{M}, \mu)$ if and only if*

$$\int_Y \left(\sum_{n\geqslant 1} \left|f_n(y)\right|^2\right)^{\frac{p}{2}} \mathrm{d}\mu(y) < +\infty.$$

Putting this together with the Minkowski inequalities, one easily deduces the following fact, which is usually stated as "$L^p(Y, \mathcal{M}, \mu)$ has type 2 and cotype p for $p \geqslant 2$": there exists a positive constant C such that for every N-tuple $(x_1, \ldots, x_N)$ of vectors of X,

$$\frac{1}{C}\left(\sum_{n=1}^{N} \|x_n\|^p\right)^{\frac{1}{p}} \leqslant \left(\int_\Omega \left\|\sum_{n=1}^{N} \varepsilon_n(\omega)x_n\right\|^2 \mathrm{d}\mathbb{P}(\omega)\right)^{\frac{1}{2}} \leqslant C\left(\sum_{n=1}^{N} \|x_n\|^2\right)^{\frac{1}{2}}.$$

When the space X has finite cotype, the series $\sum_{n\geqslant 1} \varepsilon_n(\omega)x_n$ converges a.s. in X if and only if the series $\sum_{n\geqslant 1} g_n(\omega)x_n$ converges a.s. in X, according to a result of Maurey and Pisier [4]. As a consequence, since $L^p(Y, \mathcal{M}, \mu)$ has cotype p for $p \geqslant 2$, Theorem 1.2 remains true if instead of considering the Rademacher random series $\sum_{n\geqslant 1} \varepsilon_n(\omega)c_n e_n$ we consider the gaussian random series $\sum_{n\geqslant 1} g_n(\omega)c_n e_n$.

We refer the reader to one of the books [2] or [3] for more on these topics.

Fact 2.1 gives a deterministic way to compute the critical exponent of a sequence, and the fact that $L^p(Y, \mathcal{M}, \mu)$ is of type 2 and cotype p for $p \geqslant 2$ yields some more tractable estimates for this exponent:

Fact 2.2. *Let $c = (c_n)_{n\geqslant 1}$ be a sequence of $\ell^2(\mathbb{N})$, $(e_n)_{n\geqslant 1}$ an orthonormal basis of $L^2(Y, \mathcal{M}, \mu)$. For any $p \geqslant 2$, we have that $\sum \varepsilon_n(\omega)c_n e_n$ converges a.s. if and only if*

$$\int_Y \left(\sum_{n\geqslant 1} |c_n|^2 \left|e_n(y)\right|^2\right)^{\frac{p}{2}} \mathrm{d}\mu(y)$$

is finite, and thus

(i) *if the series $\sum_{n\geqslant 1} |c_n|^2 \|e_n\|^2_{L^p(Y,\mathcal{M},\mu)}$ is convergent, then $p \leqslant p_{cr}(c)$;*
(ii) *if the series $\sum_{n\geqslant 1} |c_n|^p \|e_n\|^p_{L^p(Y,\mathcal{M},\mu)}$ is divergent, then $p \geqslant p_{cr}(c)$.*

The estimations given in (i) and (ii) above are often not sufficient to obtain the exact value of the critical exponent, and some specific properties of the functions e_n have to be used as well. We state one last general fact, implicit in [1], which can sometimes yield an interesting upper bound for $p_{cr}(c)$ when the functions $|e_n|$ concentrate near the origin in a suitable way:

Fact 2.3. *Let $(\alpha_n)_{n\geqslant 1}$ be a sequence of positive numbers. For $N \geqslant 1$, denote by Y_N the set $Y_N = \{y \in Y;\ \text{for every}\ n = 1, \ldots, N, |e_n(y)| \geqslant \alpha_n\}$. If p is such that*

$$\limsup_{N\to+\infty} \left\{ \left(\sum_{n=1}^N |c_n|^2 \alpha_n^2\right)^{\frac{p}{2}} \mu(Y_N) \right\} = +\infty,$$

then $p \geqslant p_{cr}(c)$.

Fact 2.3 follows directly from Fact 2.1:

$$\int_{Y_N} \left(\sum_{n=1}^N |c_n|^2 \left|e_n(y)\right|^2\right)^{\frac{p}{2}} \mathrm{d}\mu(y) \geqslant \int_{Y_N} \left(\sum_{n=1}^N |c_n|^2 \alpha_n^2\right)^{\frac{p}{2}} \mathrm{d}\mu(y) = \left(\sum_{n=1}^N |c_n|^2 \alpha_n^2\right)^{\frac{p}{2}} \mu(Y_N).$$

3. Proof of Theorem 1.2

Let $c = (c_n)_{n\geqslant 1} \in \ell^2(\mathbb{N})$. We first prove the upper bound $p_{cr}(c) \leqslant \frac{2d}{\alpha_*(c)}$. The argument is almost the same as in [1, Lemma 2.6], where some estimates on the Bessel functions and their zeroes show the existence of two positive constants C and γ such that for any $r \in [0, \gamma/n]$, $|e_{n,d}(r)| \geqslant Cn^{\frac{d-1}{2}}$. If $\alpha_n = Cn^{\frac{d-1}{2}}$, this shows that the measure of the set $\Omega_N = \{r \in [0,1];\ \text{for every}\ n = 1, \ldots, N,\ |e_{n,d}(r)| \geqslant \alpha_n\}$ is larger than

$$\int_0^{\frac{\gamma}{N}} r^{d-1}\,\mathrm{d}r = \frac{\gamma^d}{d\,N^d}.$$

Hence by Fact 2.3, $p \geqslant p_{cr}(c)$ if

$$\limsup_{N\to+\infty} \left(\sum_{n=1}^N n^{d-1}|c_n|^2\right) \frac{1}{N^{\frac{2d}{p}}} = +\infty.$$

It follows from (1) that this is the case when $\frac{2d}{p} < \alpha_*(c)$, which yields the bound $p_{cr}(c) \leqslant \frac{2d}{\alpha_*(c)}$.

In order to derive the lower bound $p_{cr}(c) \geqslant \frac{2d}{\alpha_*(c)}$, we will use some upper bounds on the Bessel functions obtained in [1] in order to show that if $p < \frac{2d}{\alpha_*(c)}$, the series

$$\sum_{n\geqslant 1} |c_n|^2 \|e_{n,d}\|^2_{L^p([0,1],r^{d-1}\,\mathrm{d}r)}$$

is convergent.

First of all if $p < \frac{2d}{d-1}$, the L^p-norms of the functions $e_{n,d}$ are uniformly bounded by [1, Lemma 2.5], and so the series $\sum |c_n|^2 \|e_{n,d}\|^2_{L^p}$ is obviously convergent. So let us suppose that $\frac{2d}{d-1} < p < \frac{2d}{\alpha}$ for some $\alpha > \alpha_*(c)$. Since $\|e_{n,d}\|_{L^p} = O(n^{-\frac{d}{p}+\frac{d-1}{2}})$ for $p > \frac{2d}{d-1}$ by [1, Lemma 2.5], there exists a positive constant C such that for every $N \geqslant 1$,

$$\sum_{n=1}^{N} |c_n|^2 \|e_{n,d}\|_{L^p}^2 \leqslant C \sum_{n=1}^{N} |c_n|^2 n^{d-1} n^{-\frac{2d}{p}}.$$

Since $\alpha > \alpha_*(c)$, there exists a positive constant C' such that

$$S_N = \sum_{n=1}^{N} |c_n|^2 n^{d-1} \leqslant C' N^{\alpha}$$

for any $N \geqslant 1$. Now an Abel summation by parts shows that for some positive constant C'' we have

$$\sum_{n=1}^{N} |c_n|^2 n^{d-1} n^{-\frac{2d}{p}} = |c_1|^2 + \sum_{n=2}^{N} (S_n - S_{n-1}) n^{-\frac{2d}{p}} \leqslant |c_1|^2 + C'' \sum_{n=2}^{N} \frac{1}{n^{1+\frac{2d}{p}-\alpha}}.$$

As $\frac{2d}{p} > \alpha$, this shows that the series $\sum |c_n|^2 \|e_{n,d}\|_{L^p}^2$ is convergent. Hence by Fact 2.2 we have $p_{cr}(c) \geqslant \frac{2d}{\alpha_*(c)}$, and Theorem 1.2 is proved.

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