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Optimal control

Carleman inequalities for the heat equation in singular domains

Inégalités de Carleman pour l'équation de la chaleur dans des domaines singuliers

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ABSTRACT

We consider the Cauchy problem associated to the heat equation firstly in a plane domain with a reentrant corner, then in a cracked domain. By constructing a weight function, we show a result of null controllability using Carleman estimates.

RÉSUMÉ

On considère le problème de Cauchy associé à l'équation de la chaleur dans un domaine plan avec un coin rentrant puis dans un domaine fissuré. En construisant une fonction poids, on montre un résultat de nulle contrôlabilité grâce à des estimations de type Carleman.

Version française abrégée

Notations et hypothèses

Soit Ω un domaine borné de frontière Γ, on note ν la normale unitaire sortante. Soit $\omega \Subset \Omega$ un ouvert non vide de Ω. Pour tout $T > 0$, on pose $Q_T = \Omega \times (0, T)$ et $\Sigma_T = \Gamma \times (0, T)$.

On s'intéresse aux cas suivants :

Cas 1. Ω est un domaine avec un coin rentrant en un point S de Γ d'angle φ, $\pi < \varphi < 2\pi$. $\Gamma \backslash \{S\}$ est supposée régulière.

Cas 2. Ω est un domaine avec une seule fissure rectiligne σ débouchante. On désignera par S sa pointe et par Γ_1 la partie $\Gamma \backslash \sigma$, Γ_1 est supposée régulière.

Étant donnée f dans $L^2(\Omega)$, l'unique solution $u \in H_0^1(\Omega)$ du problème de Dirichlet

$$\begin{cases} -\Delta u = f & \text{dans } \Omega, \\ u = 0 & \text{sur } \Gamma \end{cases}$$

est donnée (P. Grisvard [4]) par :

$$u(r, \theta) = u_R + c r^{\frac{\pi}{\varphi}} \sin\left(\frac{\pi}{\varphi}\theta\right), \quad \pi < \varphi \leqslant 2\pi$$

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où $u_R \in H_0^1(\Omega) \cap H^2(\Omega)$ désigne la partie régulière de u, c une constante dépendante de f et (r, θ) les coordonnées polaires locales au point S.

Notons que $D(-\Delta) \subset H^s(\Omega)$ où $\frac{3}{2} < s < 2$ dans le 1er cas, et $1 < s < \frac{3}{2}$ dans le 2éme cas.

Position du problème

Le problème de nulle contrôlabilité pour l'équation de la chaleur consiste à trouver, pour toute donnée $u_0 \in L^2(\Omega)$, un contrôle v défini dans $L^2(Q_T)$ tel que la solution du problème de Cauchy

$$(\mathbf{P}) \quad \begin{cases} u_t - \Delta u = \chi_\omega v & \text{dans } Q_T, \\ u = 0 & \text{dans } \Sigma_T, \\ u(0) = u_0 & \text{dans } \Omega \end{cases}$$

satisfait $u(T) = 0$. Cela peut être établi à partir d'une inégalité d'observabilité pour le problème adjoint associé

$$(\mathbf{P}^*) \quad \begin{cases} -q_t - \Delta q = 0 & \text{dans } Q_T, \\ q = 0 & \text{dans } \Sigma_T, \\ q(T) = q_T & \text{dans } \Omega \end{cases}$$

de type

$$\|q(0)\|^2_{L^2(\Omega)} \leqslant C \int_{\omega \times (0,T)} |q(t,x)|^2 \, dx \, dt, \quad \forall q_T \in L^2(\Omega).$$

Les estimations de Carleman sont un outil efficace pour montrer de telles inégalités. La preuve est basée sur la construction, pour tout ouvert $\omega \Subset \Omega$ non vide, d'une fonction poids vérifiant

$$\beta \in C^2(\overline{\Omega}), \quad \beta > 0 \quad \text{dans } \Omega, \quad \beta|_\Gamma = 0, \quad |\nabla \beta| \geqslant C > 0 \quad \text{dans } \overline{\Omega \backslash \omega}.$$

L'existence d'une telle fonction dans un domaine singulier n'a pas été démontrée. Néanmoins, on a pu justifier dans ce travail l'inégalité de Carleman, en construisant une fonction poids vérifiant :

$$\beta \in C^1(\overline{\Omega}) \cap W^{2,\infty}(\Omega), \quad \beta > 0 \quad \text{dans } \Omega, \quad |\nabla \beta| \geqslant C > 0 \quad \text{dans } \overline{\Omega \backslash \omega} \quad \text{et} \quad \frac{\partial \beta}{\partial \nu} \leqslant 0 \quad \text{sur } \Gamma.$$

Pour $q_T \in L^2(\Omega)$, (P^*) admet une unique solution dans $C^0([0,T], L^2(\Omega)) \cap C^0([0,T[, D(-\Delta)) \cap C^1([0,T[, L^2(\Omega))$. On rappelle que si le domaine Ω est régulier on a $D(-\Delta) = H^2(\Omega) \cap H_0^1(\Omega)$, la solution est donc assez régulière pour justifier les calculs. Ce n'est pas notre cas. Pour contourner la difficulté, on approche la solution q par une suite de fonctions régulières.

Résultat principal

Théorème. *Pour tout u_0 dans $L^2(\Omega)$, il existe $v \in L^2(Q_T)$ tel que la solution du problème* (**P**) *vérifie $u(T) = 0$.*

Dans la preuve, on obtient une inégalité grâce à la proposition suivante :

Proposition. *Il existe trois constantes $\lambda_1 = \lambda_1(\Omega, \omega) \geqslant 1$, $s_1 = s_1(\Omega, \omega)(T + T^2)$ et $C_1 = C_1(\Omega, \omega)$ telles que, pour tout $\lambda \geqslant \lambda_1$ et $s \geqslant s_1$ l'inégalité suivante soit vérifiée :*

$$\begin{aligned} &s^{-1} \int_{Q_T} e^{-2s\alpha} \xi^{-1} \left(|q_t|^2 + |\Delta q|^2\right) dx \, dt + s\lambda^2 \int_{Q_T} e^{-2s\alpha} \xi |\nabla q|^2 \, dx \, dt \\ &\quad + s^3 \lambda^4 \int_{Q_T} e^{-2s\alpha} \xi^3 |q|^2 \, dx \, dt \leqslant C_1 \left(s^3 \lambda^4 \int_{\omega \times (0,T)} e^{-2s\alpha} \xi^3 |q|^2 \, dx \, dt \right), \end{aligned} \tag{1}$$

où q est la solution de (**P***), *α et ξ sont données par* (2).

Pour cela, on construit une fonction poids qui vérifie les propriétés données par le lemme suivant :

Lemme. *Soit $\omega \Subset \Omega$ un ouvert non vide, il existe une fonction β définie dans $\overline{\Omega}$ vérifiant :*

$$\beta \in C^1(\overline{\Omega}) \cap W^{2,\infty}(\Omega), \quad \beta > 0 \quad \textit{dans } \Omega, \quad |\nabla \beta| > 0 \quad \textit{dans } \overline{\Omega \backslash \omega}, \quad \frac{\partial \beta}{\partial \nu} \leqslant 0 \quad \textit{sur } \Gamma.$$

1. Notations and assumptions

Let Ω be a bounded domain in $\mathbb{R}^2$ with boundary Γ. We denote by ν the outward unit normal vector field on Γ.
Let $\omega \Subset \Omega$ be a nonempty open subset of Ω. For $T > 0$, we set $Q_T = \Omega \times (0, T)$ and $\Sigma_T = \Gamma \times (0, T)$.
There are two different situations that will be considered in this Note:

Case 1. Ω is supposed to have a reentrant corner at a point S $(S \in \Gamma)$ of measure φ, $\pi < \varphi < 2\pi$. $\Gamma \backslash \{S\}$ is supposed regular.

Case 2. Ω contains one straight emerging crack σ, we will designate by S its tip and by Γ_1 the part $\Gamma \setminus \sigma$, Γ_1 is supposed regular.

Given f in $L^2(\Omega)$, the unique solution $u \in H_0^1(\Omega)$ of the Dirichlet problem

$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ u = 0 & \text{on } \Gamma. \end{cases}$$

is, according to P. Grisvard [4], given by

$$u(r, \theta) = u_R + c r^{\frac{\pi}{\varphi}} \sin\left(\frac{\pi}{\varphi}\theta\right), \quad \pi < \varphi \leqslant 2\pi,$$

where $u_R \in H_0^1(\Omega) \cap H^2(\Omega)$ is the regular part of u, c is a real constant depending of the data f and (r, θ) are the local polar coordinates at S.
Note that, $D(-\Delta) \subset H^s(\Omega)$ with $\frac{3}{2} < s < 2$ in Case 1 and $1 < s < \frac{3}{2}$ in Case 2.

2. Statement of the problem

The null controllability for the heat equation is as follows: For any Cauchy data u_0, find a control v defined in $L^2(Q_T)$ such that the solution of the Cauchy problem,

$$(\mathbf{P}) \quad \begin{cases} u_t - \Delta u = \chi_\omega v & \text{in } Q_T, \\ u = 0 & \text{on } \Sigma_T, \\ u(0) = u_0 & \text{in } \Omega \end{cases}$$

satisfies $u(T) = 0$. The null controllability can be deduced from an inequality of observability to the associated adjoint problem,

$$(\mathbf{P}^*) \quad \begin{cases} -q_t - \Delta q = 0 & \text{in } Q_T, \\ q = 0 & \text{on } \Sigma_T, \\ q(T) = q_T & \text{in } \Omega \end{cases}$$

of the type

$$\|q(0)\|^2_{L^2(\Omega)} \leqslant C \int\limits_{\omega \times (0,T)} |q(t, x)|^2 \, dt \, dx, \quad \forall q_T \in L^2(\Omega).$$

A powerful tool to prove such inequalities is Carleman estimates. The proof is based on the construction, for each nonempty open subset $\omega \Subset \Omega$, of a weight function β satisfying

$$\beta \in C^2(\overline{\Omega}), \quad \beta > 0 \quad \text{in } \Omega, \quad \beta|_\Gamma = 0, \quad |\nabla \beta| \geqslant C > 0 \quad \text{in } \overline{\Omega \backslash \omega}.$$

The existence of such a function in a singular domain has not been proved. Nevertheless, we were able in this work to justify Carleman's inequality by constructing a weight function β satisfying

$$\beta \in C^1(\overline{\Omega}) \cap W^{2,\infty}(\Omega), \quad \beta > 0 \quad \text{in } \Omega, \quad |\nabla \beta| \geqslant C > 0 \quad \text{in } \overline{\Omega \backslash \omega} \quad \text{and} \quad \frac{\partial \beta}{\partial \nu} \leqslant 0 \quad \text{on } \Gamma.$$

For $q_T \in L^2(\Omega)$, (P^*) admits a unique solution in

$$C^0\big([0, T], L^2(\Omega)\big) \cap C^0\big([0, T[, D(-\Delta)\big) \cap C^1\big([0, T[, L^2(\Omega)\big).$$

When the domain Ω is regular $D(-\Delta) = H^2(\Omega) \cap H_0^1(\Omega)$, the solution is then sufficiently smooth to justify the calculus. That is not our case. To encounter this difficulty, we approximate the solution q by a sequence of regular functions.

3. Main results

Theorem. *For every $u_0 \in L^2(\Omega)$, there exists $v \in L^2(Q_T)$ such that the solution u of* (**P**) *satisfies $u(T) = 0$.*

Proof of the theorem. It's a consequence of the proposition below.

First, let λ be a sufficiently large positive constant that depends only on Ω and ω. Following [2] and [3] we introduce the functions

$$\alpha(x,t) = \frac{e^{2\lambda m\|\beta\|_\infty} - e^{\lambda(m\|\beta\|_\infty + \beta(x))}}{t(T-t)}, \qquad \xi(x,t) = \frac{e^{\lambda(m\|\beta\|_\infty + \beta(x))}}{t(T-t)} \quad t \in (0,T) \tag{2}$$

where $m > 1$ and β is a suitable weight function which will be constructed in the sequel.

Proposition. *There exist three constants $\lambda_1 = \lambda_1(\Omega,\omega) \geqslant 1$, $s_1 = s_1(\Omega,\omega)(T+T^2)$ and $C_1 = C_1(\Omega,\omega)$ such that, for any $\lambda \geqslant \lambda_1$ et $s \geqslant s_1$ the next inequality holds*:

$$\begin{aligned} &s^{-1}\int\limits_{Q_T} e^{-2s\alpha}\xi^{-1}\big(|q_t|^2 + |\Delta q|^2\big)\,\mathrm{d}x\,\mathrm{d}t + s\lambda^2\int\limits_{Q_T} e^{-2s\alpha}\xi|\nabla q|^2\,\mathrm{d}x\,\mathrm{d}t + s^3\lambda^4\int\limits_{Q_T} e^{-2s\alpha}\xi^3|q|^2\,\mathrm{d}x\,\mathrm{d}t \\ &\quad\leqslant C_1\Bigg(s^3\lambda^4\int\limits_{\omega\times(0,T)} e^{-2s\alpha}\xi^3|q|^2\,\mathrm{d}x\,\mathrm{d}t\Bigg), \end{aligned} \tag{3}$$

where q is the solution of ($\mathbf{P}^*$).

We set $\psi = e^{-s\alpha}q$, we write the equation verified by ψ in the form

$$M_1\psi + M_2\psi = g_{s,\lambda}$$

where

$$\begin{cases} M_1\psi = -2s\lambda^2|\nabla\beta|^2\xi\psi - 2s\lambda\nabla\xi.\nabla\psi + \psi_t, \\ M_2\psi = s^2\lambda^2|\nabla\beta|^2\xi^2\psi + \Delta\psi + s\alpha_t\psi, \\ g_{s,\lambda} = s\lambda\Delta\beta\xi\psi - s\lambda^2|\nabla\beta|^2\xi\psi. \end{cases}$$

We note by $(M_i\psi)_j$ the j-th term in the expression of $M_i\psi$. We have:

$$\|M_1\psi\|^2_{L^2(Q_T)} + \|M_2\psi\|^2_{L^2(Q_T)} + 2\sum_{i,j=1}^{3}\big\langle (M_1\psi)_i, (M_2\psi)_j\big\rangle_{L^2(Q_T)} = \|g_{s,\lambda}\|^2_{L^2(Q_T)}.$$

The regularity of the solution q is not sufficient to do some integrations by parts in $\langle (M_1\psi)_i, (M_2\psi)_j\rangle_{L^2(Q_T)}$. We show that there exists a sequence $(q_n^T)_n \subset D((-\Delta)^2)$ which converges to q_T in $L^2(\Omega)$.

Let q_n be the solution of the approximate problem:

$$\begin{cases} \partial_t q_n + \Delta q_n = 0 & \text{in } Q_T, \\ q_n = 0 & \text{on } \Sigma_T, \\ q_n(T) = q_n^0 & \text{in } \Omega. \end{cases}$$

We set $\psi_n = e^{-s\alpha}q_n$.

Lemma 1. *We have*

$$\begin{aligned} &s^{-1}\int\limits_{Q_T} \xi^{-1}\big(|\psi_{nt}|^2 + |\Delta\psi_n|^2\big)\,\mathrm{d}x\,\mathrm{d}t + s\lambda^2\int\limits_{q} \xi|\nabla\psi_n|^2\,\mathrm{d}x\,\mathrm{d}t + s^3\lambda^4\int\limits_{q} \xi^3|\psi_n|^2\,\mathrm{d}x\,\mathrm{d}t \\ &\quad\leqslant C_1 s^3\lambda^4\int\limits_{\mathbb{O}\times(0,T)} \xi^3|\psi_n|^2\,\mathrm{d}x\,\mathrm{d}t, \end{aligned} \tag{4}$$

for $\lambda > C(\Omega,\omega)$ and $s > C(\Omega,\mathcal{O})(T+T^2)$.

Proof of Lemma 1.

Case 1. When Ω is a domain with a reentrant corner, we have just to follow [2].

Case 2. Ω is a domain with a straight emerging crack of a tip S. Suppose that S is the origin of the Euclidean coordinates system (x_1, x_2) and that the positive x_1-axis contains the crack.

Set $\Omega_\varepsilon = \Omega \backslash B(0, \varepsilon)$, $\varepsilon > 0$. By integration by parts in Ω_ε, for $\varepsilon > 0$ small enough and taking the limit as $\varepsilon \to 0$, we get the inequality (4). □

Lemma 2. *We have*

1. (q_n) *converges to* q *in* $L^2(0, T, H_0^1(\Omega))$.
2. $(\Delta\psi_n)$ *converges to* $\Delta\psi$ *in* $L^2(Q_T)$.

Letting $n \to +\infty$, *we obtain the same inequality for* ψ. *Arguing as* [2], *we deduce the Carleman's estimate* (3).

Lemma 3 *(Case 1). Let* $\omega \Subset \Omega$, *there exists a function* β *defined in* $\overline{\Omega}$ *satisfying*:

$$\beta \in C^1(\overline{\Omega}) \cap W^{2,\infty}(\Omega), \quad \beta > 0 \quad \text{in } \Omega, \quad |\nabla\beta| \neq 0 \quad \text{in } \overline{\Omega \backslash \omega} \quad \text{and} \quad \frac{\partial\beta}{\partial\nu} < 0 \quad \text{on } \Gamma.$$

Lemma 4 *(Case 2). Let* $\omega \Subset \Omega$, *there exists a function* β *defined in* $\overline{\Omega}$ *satisfying*:

$$\beta \in C^1(\overline{\Omega}) \cap W^{2,\infty}(\Omega), \quad \beta > 0 \quad \text{in } \Omega, \quad |\nabla\beta| \neq 0 \quad \text{in } \overline{\Omega \backslash \omega}, \quad \frac{\partial\beta}{\partial\nu} < 0 \quad \text{on } \Gamma_1 \quad \text{and} \quad \frac{\partial\beta}{\partial\nu^\pm} = 0 \quad \text{on } \sigma.$$

Proof of Lemma 3. The boundary Γ will be parameterized by the arc length l with origin at S.

Let us fix two points M_1 and M_2 on Γ close enough to S satisfying $l(M_1)l(M_2) < 0$. We note Γ_s the part $\widehat{M_1M_2}$ of Γ containing S and $\Gamma' = \Gamma \backslash \Gamma_s$. We can prove the existence of a function β_s satisfying

$$\beta_s \in C^2(\overline{\Omega}), \quad \beta_s > 0 \quad \text{in } \overline{\Omega}, \quad \frac{\partial\beta_s}{\partial\nu} < 0 \quad \text{on } \Gamma_s \quad \text{and} \quad |\nabla\beta_s| > 0 \quad \text{in } \Omega.$$

One can take, for instance

$$\beta_s(x, y) = by + c$$

where $b > 0$ and $c > 0$ is large enough.

We denote by $\Omega' \subset \Omega$ a smooth subdomain of class C^2 such that $\partial\Omega' = \Gamma_0 \cup \Gamma'$ and $\frac{\partial\beta_s}{\partial\nu} < 0$ on Γ_0.

Let us now set $g_0 = \beta_s|_{\Gamma_0}$ and $h_0 = \frac{\partial\beta_s}{\partial\nu}|_{\Gamma_0}$, we extend g_0 (resp. h_0) to $\partial\Omega'$ by a function g (resp. h) of class C^2 (resp. C^1) such that $g > 0$, $h < 0$ on $\partial\Omega'$.

Let $\widetilde{g}$ a C^2 lift of g_0 to Ω' such that $\frac{\partial\widetilde{g}}{\partial\nu} = h$ on $\partial\Omega'$ and $\widetilde{g} > 0$ in Ω'.

For $\varepsilon > 0$, we set $U_\varepsilon = \{X \in \Omega', dist(X, \partial\Omega') < \varepsilon\}$. As $\frac{\partial\widetilde{g}}{\partial\nu} = h < 0$ on $\partial\Omega'$ then, for $\varepsilon > 0$ small enough we have $|\nabla\widetilde{g}| \neq 0$ in $\overline{U_\varepsilon}$, so that $\widetilde{g}$ satisfies: $\widetilde{g} > 0$ in $\overline{U_\varepsilon}$, $|\nabla\widetilde{g}| \neq 0$ in $\overline{U_\varepsilon}$ and $\widetilde{g} \in C^2(\overline{U_\varepsilon})$.

Using the same ideas as in [1] and [3], we establish the existence of a function β' satisfying

$$\beta' \in C^2(\overline{\Omega'}), \quad \beta' > 0 \quad \text{in } \Omega', \quad |\nabla\beta'| \neq 0 \quad \text{in } \overline{\Omega' \backslash \omega} \quad \text{and} \quad \beta' = g \quad \text{on } \partial\Omega'.$$

To conclude, we take

$$\beta = \begin{cases} \beta_s & \text{in } \overline{\Omega/\Omega'}, \\ \beta' & \text{in } \overline{\Omega'}. \end{cases} \qquad \square$$

Proof of Lemma 4. Recall that $\Gamma = \sigma \cup \Gamma_1$.

Set $\sigma = [S; S']$, $S' \in \Gamma_1$ and consider two points M_1, M_2 of Γ_1 close enough to S' such that $S' \in \widehat{M_1M_2}$. We denote this arc by Γ'' and set $\Gamma_1 = \Gamma' \cup \Gamma''$.

We construct a function β_s satisfying

$$\beta_S \in C^2(\overline{\Omega}), \quad \beta_s > 0 \quad \text{in } \overline{\Omega}, \quad \frac{\partial\beta_s}{\partial\nu} < 0 \quad \text{on } \Gamma'', \quad \frac{\partial\beta_s}{\partial\nu_\pm} = 0 \quad \text{on } \sigma \quad \text{and} \quad |\nabla\beta_s| > 0 \quad \text{in } \Omega.$$

We choose a C^2 open subset $\Omega' \subset \Omega$ such that

$$\partial\Omega' = \Gamma_0 \cup \Gamma' \quad \text{and} \quad \frac{\partial\beta_s}{\partial\nu} < 0 \quad \text{on } \Gamma_0$$

and we follow the proof as in Case 1. □

Remark. We generalize the results etablished in Case 1 to any polygon.

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