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Lie Algebras/Geometry

An algebra of observables for cross ratios ☆

Une algèbre d'observables pour les birapports

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ABSTRACT

We define a Poisson Algebra called the *swapping algebra* using the intersection of curves in the disk. We interpret a subalgebra of the fraction swapping algebra – called the *algebra of multifractions* – as an algebra of functions on the space of cross ratios and thus as an algebra of functions on the Hitchin component as well as on the space of $\mathrm{SL}(n,\mathbb{R})$-opers with trivial holonomy. We finally relate our Poisson structure to the Drinfel'd–Sokolov structure and to the Atiyah–Bott–Goldman symplectic structure.

RÉSUMÉ

Nous introduisons une algèbre de Poisson, l'*algèbre d'échange*, définie à l'aide de l'intersection des courbes dans le disque. Nous interprétons l'*algèbre des multifractions* – une sous-algèbre de l'algèbre des fractions de l'algèbre d'échange – comme une algèbre de fonctions sur l'espace des birapports et donc en particulier comme une algèbre de fonctions sur la composante de Hitchin ainsi que sur l'espace des $\mathrm{SL}(n,\mathbb{R})$-opers d'holonomie triviale. Nous relions alors notre structure de Poisson à la structure de Poisson de Drinfel'd–Sokolov ainsi qu'à la structure symplectique d'Atiyah–Bott–Goldman.

Version française abrégée

Si (X,x,Y,y) est un quadruplet de points distincts du cercle, l'*intersection* $\mathfrak{I}(X,x,Y,y)$ des couples (X,x) et (Y,y) est l'intersection dans le disque des deux courbes orientées joignant respectivement X à x et Y à y. Cette intersection s'étend à tous les couples de points (voir formule 2) en prenant sa valeur dans $\{-1,-1/2,0,1/2,1\}$. Nous noterons désormais Xx le couple de points (X,x).

Soit $\mathcal{P}$ un sous-ensemble de points du cercle. L'*algèbre d'échange* $\mathcal{Z}(\mathcal{P})$ est l'algèbre associative commutative – c'est-à-dire l'algèbre polynomiale – engendrée sur $\mathbb{Q}$ par les couples Xx avec les relations $Xx=0$ si $X=x$, où X et x appartiennent à $\mathcal{P}$. On définit le *crochet d'échange de couples* sur les générateurs par

$$\{Xx, Yy\} = \mathfrak{I}(X,x,Y,y)Xy.Yx. \tag{1}$$

On étend ce crochet à toute l'algèbre $\mathcal{Z}(\mathcal{P})$ de façon à ce que $u \to \{u,v\}$ et $u \to \{v,u\}$ soient des dérivations pour tout v.

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Notre premier résultat – Théorème 1 – est que cette algèbre est une algèbre de Poisson. Notre but dans cette note est d'annoncer deux résultats reliant cette algèbre de Poisson à deux structures symplectiques connues

- La structure symplectique de Drinfel'd–Sokolov sur les $\mathrm{SL}(n,\mathbb{R})$-opers [9,2].
- La structure symplectique d'Atiyah–Bott–Goldman sur la variété des caractères des représentations d'un groupe de surface orientée dans $\mathrm{SL}(n,\mathbb{R})$ [1,4].

Une telle relation avait été prévue par Witten dans [10]. Nous allons relier ces différentes structures grâce à la notion de birapport utilisée dans [6,7].

Rappelons qu'un *birapport faible* b sur $\mathcal{P}$ est une fonction à valeurs réelles définie sur $\mathcal{P}^{4*} := \{(x,y,z,t) \in \mathcal{P}^4 \mid x \neq t \text{ et } y \neq z\}$ et vérifiant les relations

$$x=y \quad \text{ou} \quad z=t \quad \Rightarrow \quad \mathbf{b}(x,y,z,t)=0, \qquad x=z \quad \text{ou} \quad y=t \quad \Rightarrow \quad \mathbf{b}(x,y,z,t)=1,$$

$$\mathbf{b}(x,y,z,t)=\mathbf{b}(x,y,w,t)\mathbf{b}(w,y,z,t), \qquad \mathbf{b}(x,y,z,t)=\mathbf{b}(x,y,z,w)\mathbf{b}(x,w,z,t).$$

Une *bifraction* est un élément de l'algèbre des fractions de $\mathcal{Z}(\mathcal{P})$ de la forme

$$[X;x;Y;y] := \frac{Xy.Yx}{Xx.Yy}.$$

L'*algèbre des multifractions* est la sous-algèbre commutative $\mathcal{B}(\mathcal{P})$ de l'algèbre des fractions de $\mathcal{Z}(\mathcal{P})$ engendrée par les bifractions, c'est aussi le sous-espace vectoriel engendré par les expressions de la forme (4). L'algèbre des multifractions est stable par l'extension du crochet d'échange de couples. Une bifraction donne naissance à une fonction sur l'espace $\mathbb{B}(\mathcal{P})$ des birapports sur $\mathcal{P}$: la valeur de la bifraction $[X;x;Y;y]$ pour le birapport $\mathbf{b}$ est donnée par

$$[X;x;Y;y](\mathbf{b}) := \mathbf{b}(X,x,Y,y).$$

Cette application s'étend en un morphisme de l'algèbre des multifractions dans l'algèbre des fonctions sur $\mathbb{B}(\mathcal{P})$.

Rappelons maintenant que l'espace des $\mathrm{SL}(n,\mathbb{R})$-opers d'holonomie triviale et la composante de Hitchin (voir [5]) de $\mathrm{Rep}(\pi_1(S),\mathrm{PSL}(n,\mathbb{R}))$ s'interprètent tous les deux comme des sous-espaces de $\mathbb{B}(\mathcal{P})$:

- Il est classique que les $\mathrm{SL}(n,\mathbb{R})$-opers d'holonomie triviale s'interprètent comme les *courbes de Frenet* de classe C^∞ à valeurs dans $\mathbb{RP}^{n-1}$ (voir [9,2,3]). Par ailleurs, une courbe de Frenet donne naissance à un birapport par la formule (5).
- De même d'après [6,7], les représentations de Hitchin s'interprètent comme des birapports sur le bord à l'infini $\partial_\infty\pi_1(S)$ du groupe fondamental de S.

Autrement dit, nous pouvons interpréter une multifraction à la fois comme une fonction sur l'espace des $\mathrm{SL}(n,\mathbb{R})$-opers et comme une fonction sur la composante de Hitchin. Les résultats annoncés dans cette Note sont les suivants :

(1) Nous montrons que le crochet de Poisson de Drinfel'd–Sokolov coïncide avec le crochet d'échange de couples pour les multifractions (voir Théorème 2).
(2) Nous montrons le crochet d'Atiyah–Bott–Goldman coïncide asymptotiquement avec le crochet d'échange de couples pour certaines multifractions et pour des suites particulières de sous-groupes d'indice finis de $\pi_1(S)$ (voir Théorème 3).

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1. The swapping algebra

1.1. Intersection of ordered pairs of points on the circle

We recall that, if (X,x,Y,y) is a quadruple of points of the interval $]0,1[$, the *intersection* $\mathfrak{I}(X,x,Y,y)$ of (X,x) and (Y,y) is

$$\frac{1}{2}\big(\mathrm{Sign}(X-x)\,\mathrm{Sign}(X-y)\,\mathrm{Sign}(y-x)-\mathrm{Sign}(X-x)\,\mathrm{Sign}(X-Y)\,\mathrm{Sign}(Y-x)\big), \tag{2}$$

where $\mathrm{Sign}(u)=-1,0,1$ whenever $u<0$, $u=0$ and $u>0$ respectively. Now, if (X,x,Y,y) is a quadruple of points of the oriented circle $\mathbb{T}$, we check that the intersection of (X,x,Y,y) in the interval $\mathbb{T}\setminus\{z\}$ does not depend on z if $z \notin \{X,x,Y,y\}$ and is thus declared to be the intersection of (X,x,Y,y) in $\mathbb{T}$.

When the four points (X,x,Y,y) are pairwise distinct, $\mathfrak{I}(X,x,Y,y)$ is the intersection of the oriented curves joining X to x and joining Y to y in the disk. In this case, the intersection belongs $\{-1,0,1\}$, in general the intersection belongs to $\{-1,-1/2,0,1/2,1\}$.

1.2. The Poisson swapping algebra

Let $\mathcal{P}$ be a subset of the circle. We represent an ordered pair (X, x) of points of $\mathcal{P}$ by the expression Xx. We consider the associative commutative algebra $\mathcal{Z}(\mathcal{P})$ generated over $\mathbb{Q}$ by ordered pairs of points on $\mathcal{P}$, together with the relations $Xx = 0$ when $X = x$.

Let α be any real number. The *swapping bracket* is defined on generators by

$$\{Xx, Yy\}_\alpha = \Im(X, x, Y, y)(\alpha.Xx.Yy + Xy.Yx), \tag{3}$$

and extended to $\mathcal{Z}(\mathcal{P})$ so that $u \to \{u, v\}_\alpha$ and $u \to \{v, u\}_\alpha$ are derivations. The *swapping algebra* $\mathcal{Z}(\mathcal{P})_\alpha$ is the algebra $\mathcal{Z}(\mathcal{P})$ equipped with the swapping bracket.

Theorem 1. *The bracket* $\{\,,\}_\alpha$ *satisfies the Jacobi identity. Hence, the algebra* $\mathcal{Z}(\mathcal{P})_\alpha$ *is a Poisson Algebra.*

This theorem only uses formal properties of the intersection and can be generalised in a more abstract setting.

2. The algebra of multifractions

Let again $\mathcal{P}$ be a subset of the circle. A *cross fraction* is an element of the algebra of fractions $\mathcal{Q}(\mathcal{P})$ of $\mathcal{Z}(\mathcal{P})$ of the form

$$[X; x; Y; y] := \frac{Xy.Yx}{Xx.Yy},$$

where $X \neq x$ and $Y \neq y$. More generally, a *multifraction* is an element of $\mathcal{Q}(\mathcal{P})$ of the form

$$\frac{X_1 x_{\sigma(1)} \dots X_n x_{\sigma(n)}}{X_1 x_1 \dots X_n x_n}, \tag{4}$$

where σ is a permutation of $\{1 \dots n\}$ and for all i, $X_i \neq x_i$.

Let $\mathcal{B}(\mathcal{P})$ be the vector space generated by multifractions. Observe that $\mathcal{B}(\mathcal{P})$ is the associative commutative algebra generated by cross fractions and moreover is stable by the Poisson bracket and is thus a Poisson Algebra. Finally the restriction $\{\,,\}_W$ of the bracket $\{\,,\}_\alpha$ is independent of α.

Then, the *algebra of multifractions* is the vector space $\mathcal{B}(\mathcal{P})$ equipped with the commutative associative product and the Poisson bracket $\{\,,\}_W$.

2.1. Multifractions and cross ratios

We want to see the algebra of multifractions as an algebra of observables, that is an algebra of functions on a space. We first see the algebra of multifractions as a subalgebra of functions on the set of all cross ratios. Recall from [7] that a *weak cross ratio* on a set $\mathcal{P}$ is a real valued function $\mathbf{b}$ on $\mathcal{P}^{4*} := \{(x, y, z, t) \in \mathcal{P}^4 \mid x \neq t, \text{ and } y \neq z\}$ which satisfies the following rules

$$x = y \quad \text{or} \quad z = t \quad \Rightarrow \quad \mathbf{b}(x, y, z, t) = 0, \qquad x = z \quad \text{or} \quad y = t \quad \Rightarrow \quad \mathbf{b}(x, y, z, t) = 1,$$

$$\mathbf{b}(x, y, z, t) = \mathbf{b}(x, y, w, t)\mathbf{b}(w, y, z, t), \qquad \mathbf{b}(x, y, z, t) = \mathbf{b}(x, y, z, w)\mathbf{b}(x, w, z, t).$$

If Γ is a group acting on $\mathcal{P}$, we say that a weak cross ratio is *invariant* under Γ if it is invariant under the diagonal action. Every cross fraction on $\mathcal{P}$ defines a natural function on the set $\mathbb{B}(\mathcal{P})$ of weak cross ratios on $\mathcal{P}$ by

$$[X; x; Y; y](\mathbf{b}) := \mathbf{b}(X, x, Y, y).$$

More generally, this definition gives rise to an homomorphism of the associative algebra $\mathcal{B}(\mathcal{P})$ into the algebra of functions on $\mathbb{B}(\mathcal{P})$. Therefore, in some sense our Theorem 1 gives a Poisson structure on the set $\mathbb{B}(\mathcal{P})$.

2.2. Frenet curves and cross ratios

A curve ξ defined from the circle $\mathbb{T}$ to $\mathbf{P}(\mathbb{R}^n)$ is a *Frenet curve* if there exists a curve $(\xi^1, \xi^2, \dots, \xi^{n-1})$ defined on $\mathbb{T}$, called the *osculating flag curve*, with values in the flag variety such that for every x in $\mathbb{T}$, $\xi(x) = \xi^1(x)$, and moreover

- For every pairwise distinct points $(x_1, \dots, x_l)$ in $\mathbb{T}$ and positive integers $(n_1, \dots, n_l)$ such that $\sum_{i=1}^{i=l} n_i \leqslant n$, then the sum $\xi^{n_i}(x_i) + \dots + \xi^{n_l}(x_l)$ is direct.
- For every x in $\mathbb{T}$ and positive integers $(n_1, \dots, n_l)$ such that $p = \sum_{i=1}^{i=l} n_i \leqslant n$, then $\lim_{\substack{(y_1,\dots,y_l)\to x,\\ y_i \text{all distinct}}} (\bigoplus_{i=1}^{i=l} \xi^{n_i}(y_i)) = \xi^p(x)$.

We call ξ^{n-1} the *osculating hyperplane*.

Let ξ be a Frenet curve and ξ^* be its associated osculating hyperplane curve. The *weak cross ratio* associated to this pair of curves is the function on $\mathbb{T}^{4*}$ defined by

$$\mathbf{b}_{\xi,\xi^*}(x,y,z,t) = \frac{\langle\widehat{\xi}(x)|\widehat{\xi}^*(y)\rangle\langle\widehat{\xi}(z)|\widehat{\xi}^*(t)\rangle}{\langle\widehat{\xi}(z)|\widehat{\xi}^*(y)\rangle\langle\widehat{\xi}(x)|\widehat{\xi}^*(t)\rangle}, \tag{5}$$

where for every u, we choose an arbitrary nonzero vector $\widehat{\xi}(u)$ and $\widehat{\xi}^*(u)$ respectively in $\xi(u)$ and $\xi^*(u)$.

3. Two incarnations of the algebra of multifractions

Our aim now is to relate the Poisson structure on $\mathbb{B}(\mathcal{P})$ to two classical Poisson structures namely

- the Drinfel'd–Sokolov structure on the space of $\mathrm{SL}(n,\mathbb{R})$-opers,
- the Atiyah–Bott–Goldman symplectic structure on the character variety of a surface group in $\mathrm{SL}(n,\mathbb{R})$.

Witten in [10] has foreshadowed a relation between these two spaces which were proved to be related in [6,7]. Our purpose here is to relate their symplectic structures.

3.1. Multifractions and opers

The space of smooth Frenet curves carries a Poisson structure from the Drinfel'd-Sokolov reduction – and is identified to the space of $\mathrm{SL}(n,\mathbb{R})$-opers with trivial holonomy – whose Poisson bracket is denoted by $\{,\}_{\mathrm{DS}}$ (see [9,2,3]). Thus, a multifraction being a function on $\mathbb{B}(\mathcal{P})$ is also function on the space of Frenet curves.

Our second theorem identifies the two Poisson brackets.

Theorem 2. *The swapping Poisson bracket coincides with the Drinfel'd–Sokolov bracket for multifractions. That is, for every multifractions b_0 and b_1,*

$$\{b_0, b_1\}_{\mathrm{DS}} = \{b_0, b_1\}_{\mathrm{W}}.$$

3.2. Multifractions and the Goldman algebra

3.2.1. The Atiyah–Bott–Goldman structure and the Goldman algebra

Let S be a closed surface. For any semi-simple Lie group G, the character variety $\mathrm{Rep}(\pi_1(S),G)$ of conjugacy classes of homomorphisms of $\pi_1(S)$ in G admits a Poisson structure (see [1,4]). When $G=\mathrm{SL}(n,\mathbb{R})$, a preferred component called the *Hitchin component* has been identified with a space of $\pi_1(S)$-invariant cross ratios on $\partial_\infty\pi_1(S)$ in [6,7]. We denote by $\mathcal{A}(S)$ the Poisson Algebra of smooth functions on the Hitchin component and $\{,\}_S$ its Poisson bracket. Since representations in the Hitchin component are cross ratios, we have a homomorphism F_S of associative algebras from $\mathcal{B}(\partial_\infty\pi_1(S))$ to $\mathcal{A}(S)$. If S_m is a finite covering of S, we also denote by R_S the restriction map from $\mathcal{A}(S_m)$ to $\mathcal{A}(S)$.

3.2.2. Coverings

Let $\mathcal{P}$ be the subset of $\partial_\infty\pi_1(S)$ which consists of fixed points of nontrivial elements of $\pi_1(S)$. Let $\mathcal{G}$ be the set of ordered pairs of points $\gamma=(\gamma^-,\gamma^+)$ in $\mathcal{P}^2$ which corresponds to fixed points by a nontrivial element of the group $\pi_1(S)$. Observe that given any finite index subgroup Γ of $\pi_1(S)$, the set $\mathcal{G}$ is in bijection with the set of primitive elements of Γ, where by definition a primitive element of Γ is an element g that is not of the form h^p with $p>1$ and $h\in\Gamma$. In the sequel, we shall freely identify elements of $\mathcal{G}$ with primitive elements in any finite index subgroup of Γ.

We say a nested sequence $\{\Gamma_m\}_{m\in\mathbb{N}}$ of finite index subgroups of $\pi_1(S)$ is *vanishing* if the following holds: let γ and η be elements in $\mathcal{G}$, let γ_m and η_m be the corresponding primitive elements in Γ_m, then there exists m_0 so that for every $m\geqslant m_0$ the geodesics corresponding to γ_m and η_m have at most one intersection point and moreover the geometric intersection is $\mathfrak{I}(\gamma^-,\gamma^+,\eta^-,\eta^+)$. It follows from the double coset separability proved by G. Niblo in [8] that vanishing sequences exist.

Observe finally that associated with a sequence $\sigma=\{\Gamma_m\}_{m\in\mathbb{N}}$ of nested finite index subgroups of $\pi_1(S)$ is the inverse limit S_σ of $S_m:=\tilde{S}/\Gamma_m$. Similarly we consider the inverse limit $\mathcal{A}(S_\sigma)$ of $\mathcal{A}(S_m)$. Then the homomorphism F_σ from $\mathcal{B}(\mathcal{P})$ to $\mathcal{A}(S_\sigma)$ is injective.

Let $\{g_m\}_{m\in\mathbb{N}}$ be a sequence of functions, so that $g_m\in\mathcal{A}(S_m)$, we say that $\{g_m\}_{m\in\mathbb{N}}$ converges to the function h in $\mathcal{A}(S_\sigma)$ and write

$$\lim_{m\to\infty} g_m = h,$$

if for any p, we have $\lim_{n\to\infty}\mathsf{R}_{S_p}(g_n)=\mathsf{R}_{S_p}(h)$.

3.2.3. Atiyah–Bott–Goldman Poisson bracket and the swapping bracket

Our third theorem relates the Atiyah–Bott–Goldman Poisson bracket $\{,\}_S$ on the Hitchin component of S with the swapping bracket.

Theorem 3. *Let $\{\Gamma_m\}_{m\in\mathbb{N}}$ be a vanishing sequence of subgroups of $\pi_1(S)$, let b_0 and b_1 be two elements of $\mathcal{B}(\mathcal{P})$, then*

$$\lim_{n\to\infty}\left\{\mathsf{F}_{S_n}(b_0),\mathsf{F}_{S_n}(b_1)\right\}_{S_n}=\mathsf{F}_\sigma\left(\{b_0,b_1\}_{\mathrm{W}}\right).$$

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