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Pointed Hopf algebras over some sporadic simple groups ☆

Algèbres de Hopf pointées sur quelques groupes simples sporadiques

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ABSTRACT

Any finite-dimensional complex pointed Hopf algebra with group of group-likes isomorphic to a sporadic group, with the possible exception of the Fischer group Fi_{22}, the Baby Monster B and the Monster M, is a group algebra.

RÉSUMÉ

Soit G un groupe sporadique différent du groupe de Fischer Fi_{22}, du bébé monstre B et du monstre M. Soit H une algèbre de Hopf complexe pointée de dimension finie dont le groupe des éléments dont le co-produit est égal au carré tensoriel est isomorphisme à G, alors H est isomorphe a l'algèbre de groupe de G.

1. Introduction

Let $\Bbbk$ be an algebraically closed field of characteristic 0. In this Note, we announce a new contribution to the classification of finite-dimensional Hopf algebras over $\Bbbk$. As is known, different classes of finite-dimensional Hopf algebras have to be studied separately because the pertaining methods are radically different. There is a method for pointed Hopf algebras (those whose coradical is a group algebra $\Bbbk G$) that has been applied with satisfactory results when G is Abelian [8]; an exposition of the method can be found in [7]. Recently, it appeared that many finite simple (or almost simple) groups G admit very few finite-dimensional, pointed Hopf algebras with coradical isomorphic to $\Bbbk G$:

- Any finite-dimensional complex pointed Hopf algebra with group of group-likes isomorphic to $\mathbb{A}_m$, $m \geqslant 5$, is a group algebra [2].
- Same for the groups $SL(2, 2^n)$, $n > 1$ [10] and M_{20}, $M_{21} = PSL(3, 4)$ [11].
- Most of the pointed Hopf algebras over the symmetric groups have infinite dimension, with the exception of a short list of open possibilities, see [2,4] and references therein. More precisely, most of the irreducible Yetter–Drinfeld modules have infinite-dimensional Nichols algebras (see below).

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This is a report on finite-dimensional pointed Hopf algebras over sporadic simple groups. As part of our results, we have the following:

Theorem 1. *Let G be any sporadic simple group, different from the Fischer group Fi_{22}, the Baby Monster B and the Monster M. If H is a finite-dimensional pointed Hopf algebra with $G(H) \simeq G$, then $H \simeq \Bbbk G$.*

The Theorem holds more generally over any field of characteristic 0, since the property of being pointed is stable under extension of scalars.

1.1. Glossary

For the reader's convenience, we recall a few definitions that are central to our work. More information can be found in [5,7]. Let H be a Hopf algebra with comultiplication Δ and bijective antipode $\mathcal{S}$.

- An element $g \neq 0$ in H is a *grouplike* if $\Delta(g) = g \otimes g$; the set of all grouplikes is a group $G(H)$ with multiplication given by the product of H.
- A *Yetter–Drinfeld module* over H is a left H-module M that bears also a structure $\lambda : M \to H \otimes M$ of H-comodule, compatible with the action in an appropriate sense. If H is finite-dimensional, then a Yetter–Drinfeld module is the same as a module over the Drinfeld double of H. For instance, if $H = \Bbbk G$ is the group algebra of a finite group G, then a Yetter–Drinfeld module over H is a left G-module M that bears also a G-gradation $M = \bigoplus_{g \in G} M_g$, compatibility meaning that $h \cdot M_g = M_{hgh^{-1}}$ for all $h, g \in G$.
- A *rack* is a pair $(X, \triangleright)$ where X is a non-empty set and $\triangleright : X \times X \to X$ is an operation such that the map $\varphi_x = x \triangleright \_\_$ is bijective for any $x \in X$, and $x \triangleright (y \triangleright z) = (x \triangleright y) \triangleright (x \triangleright z)$ for all $x, y, z \in X$. A map $q : X \times X \to GL(n, \Bbbk)$ is a *2-cocycle* of degree n if

$$q_{x, y \triangleright z} q_{y,z} = q_{x \triangleright y, x \triangleright z} q_{x,z}, \quad \text{for all } x, y, z \in X.$$

- A *braided vector space* is a pair (V, c) where V is a vector space and $c \in GL(V \otimes V)$ fulfills the braid equation: $(c \otimes \mathrm{id})(\mathrm{id} \otimes c)(c \otimes \mathrm{id}) = (\mathrm{id} \otimes c)(c \otimes \mathrm{id})(\mathrm{id} \otimes c)$. Examples:
  (i) Any Yetter–Drinfeld module is a braided vector space in a natural way.
  (ii) Let X be a finite rack, q a 2-cocycle of degree n, $V = \Bbbk X \otimes \Bbbk^n$, where $\Bbbk X$ is the vector space with basis e_x, $x \in X$. We denote $e_x v := e_x \otimes v$. Consider the linear isomorphism $c^q : V \otimes V \to V \otimes V$, $c^q(e_x v \otimes e_y w) = e_{x \triangleright y} q_{x,y}(w) \otimes e_x v$, $x, y \in X$, $v, w \in \Bbbk^n$. Then (V, c^q) is a braided vector space.

  The braided vector spaces arising as Yetter–Drinfeld modules over group algebras of finite groups can be presented in terms of racks and cocycles, see a bit more of information below.
- We assume the reader familiar with the important notion of the *Nichols algebra* of a braided vector space, discussed at length in [7]. In short, one of the possible definitions of the Nichols algebra $\mathfrak{B}(V)$ of a braided vector space (V, c) is as follows. Since c satisfies the braid equation, it induces a representation of the braid group $\mathbb{B}_n$, $\rho_n : \mathbb{B}_n \to GL(V^{\otimes n})$, for each $n \geqslant 2$. Let $Q_n = \sum_{\sigma \in \mathbb{S}_n} \rho_n(M(\sigma)) \in End(V^{\otimes n})$, where $M : \mathbb{S}_n \to \mathbb{B}_n$ is the so-called Matsumoto section (not a morphism of groups, but preserves product when length is preserved). Then $\mathfrak{B}(V)$ is the quotient of the tensor algebra $T(V)$ by $\bigoplus_{n \geqslant 2} \ker Q_n$, in fact a 2-sided ideal of $T(V)$. If c is the usual switch, then $\mathfrak{B}(V)$ is just the symmetric algebra of V; but in general the determination of a Nichols algebra is quite a difficult task.

2. Outline of the proof

A complete proof of Theorem 1 for the groups M_{22} and M_{24} is contained in [9]; the proof for the other groups is included in [3].

We sketch now the proof in two main reductions. The first one has been explained in several places, with detail in [7], but we include a brief summary for completeness. We remind that if U is a braided vector subspace of V, then $\mathfrak{B}(U) \hookrightarrow \mathfrak{B}(V)$.

2.1. A general reduction

Let G be a finite group, H a pointed Hopf algebra with $G(H) \simeq G$. Then there are two basic invariants of H, a Yetter–Drinfeld module V over $\Bbbk G$ (called the infinitesimal braiding of H) and its Nichols algebra $\mathfrak{B}(V)$. We have $|G| \dim \mathfrak{B}(V) \leqslant \dim H$. Therefore, the following statements are equivalent:

(1) If H is a finite-dimensional pointed Hopf algebra with $G(H) \simeq G$, then $H \simeq \Bbbk G$.
(2) If $V \neq 0$ is a Yetter–Drinfeld module over $\Bbbk G$, then $\dim \mathfrak{B}(V) = \infty$.
(3) If V is an *irreducible* Yetter–Drinfeld module over $\Bbbk G$, then $\dim \mathfrak{B}(V) = \infty$.

2.2. Looking at subracks

We focus on (3) above. The second reduction has been the basis of our recent papers. It starts from the well-known classification of irreducible Yetter–Drinfeld modules over $\Bbbk G$ by pairs $(\mathcal{O}, \rho)$, where $\mathcal{O}$ is a conjugacy class in G and ρ is an irreducible representation of the stabilizer G^s of a fixed point $s \in \mathcal{O}$. Now, the definition of the Nichols algebra $\mathfrak{B}(\mathcal{O}, \rho)$ of the corresponding Yetter–Drinfeld module $M(\mathcal{O}, \rho)$ just depends on the braiding. If $\dim \rho = 1$, then this braiding depends only on the *rack* $\mathcal{O}$ and a 2-cocycle $q : \mathcal{O} \times \mathcal{O} \to \Bbbk^\times$ [5]. Namely, $\mathcal{O}$ is a rack with the product $x \triangleright y := xyx^{-1}$, $M(\mathcal{O}, \rho)$ has a natural basis $(e_x)_{x \in \mathcal{O}}$ and the braiding is given by $c(e_x \otimes e_y) = q_{xy} e_{x \triangleright y} \otimes e_x$. If there exists a subrack X of $\mathcal{O}$ such that the Nichols algebra of the braided vector space defined by X and the restriction of q is infinite dimensional, then $\dim \mathfrak{B}(\mathcal{O}, \rho) = \infty$.

We recall some examples of racks which are relevant in this work.

(i) Abelian racks: those racks X such that $x \triangleright y = y$ for all $x, y \in X$.
(ii) $\mathcal{D}_p$: the class of involutions in the dihedral group $\mathbb{D}_p$ (of order $2p$), p a prime.
(iii) $\mathfrak{O}$: the class of 4-cycles in $\mathbb{S}_4$.
(iv) Doubles of racks: if X is a rack, then $X^{(2)}$ denotes the disjoint union of two copies of X each acting on the other by left multiplication.

We are interested in finding subracks which are Abelian, or isomorphic to $\mathcal{D}_p^{(2)}$ or to $\mathfrak{O}^{(2)}$, by the following reasons:

(A) If X is Abelian, then the corresponding braided vector space is of diagonal type. Braided vector spaces of diagonal type with finite-dimensional Nichols algebra where classified in [13]; thus, we just need to check if the matrix (q_{xy}) belongs or not to the list in [13].
(B) If X is isomorphic either to $\mathcal{D}_p^{(2)}$ or to $\mathfrak{O}^{(2)}$, then for some specific cocycles, the related Nichols algebras have infinite dimension [6, Ths. 4.7, 4.8].

Variations:

(a) If $\dim \rho > 1$, similar arguments apply.
(b) Sometimes the rack X is not Abelian, but the braided vector space produced by X and the 2-cocycle can be realized with an Abelian rack, by a suitable change of basis.
(c) Let $F < G$ be a subgroup, $s \in F$, $\mathcal{O}^F$, resp. $\mathcal{O}^G$ the conjugacy class of s in F, resp. in G. If $\dim \mathfrak{B}(\mathcal{O}^F, \tau) = \infty$ for any irreducible representation τ of F^s, then $\dim \mathfrak{B}(\mathcal{O}^G, \rho) = \infty$ for any irreducible representation ρ of G^s.
(d) A conjugacy class $\mathcal{O}$ is real if $\mathcal{O} = \mathcal{O}^{-1}$. It is quasireal if $\mathcal{O} = \mathcal{O}^m$ for some integer m, $1 < m < N$, where N is the order of the elements in $\mathcal{O}$. The search of subracks isomorphic to $\mathcal{D}_p^{(2)}$ or to $\mathfrak{O}^{(2)}$, as well as the verification that the restriction of the cocycle q is as needed in (2.2), is greatly simplified in a real (quasireal) conjugacy class [1].
(e) We say that a rack X is of type D if there exists a decomposable subrack $Y = R \coprod S$ of X such that $r \triangleright (s \triangleright (r \triangleright s)) \neq s$, for some $r \in R$, $s \in S$. If a conjugacy class $\mathcal{O}$ is a rack of type D, then $\dim \mathfrak{B}(\mathcal{O}, \rho) = \infty$ for any ρ (see [2] and Theorem 8.6 of [14]).

2.3. Computations

We now fix a sporadic group G as in Theorem 1. We extracted relevant information from the ATLAS [15] with the `AtlasRep` package [16]. Then, we checked when a conjugacy class is real or quasireal or of type D. We used GAP [12] for the computations.

These tools allow us to apply the techniques sketched above to all pairs $(\mathcal{O}, \rho)$ and establish the validity of (2.1).

2.4.

Some of these results were announced in several meetings:

– Hopf Algebras and Related Topics, A conference in honor of Professor Susan Montgomery. University of Southern California, Los Angeles, USA. February 2009.
– IV Encuentro Nacional de Álgebra, Córdoba, Argentina. August, 2008.
– First De Brún Workshop on Computational Algebra, National University of Ireland, Galway, Ireland. August, 2008
– Groupes quantiques dynamiques et cagories de fusion. CIRM, Luminy, France. April 2008.

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