C. R. Acad. Sci. Paris, Ser. I 348 (2010) 1123–1128

Contents lists available at ScienceDirect

C. R. Acad. Sci. Paris, Ser. I

www.sciencedirect.com

Mathematical Physics

External regions of nonlinearly perturbed Kerr spacetimes satisfying the peeling decay

Régions extérieures de l'espace-temps de Kerr perturbé par une non-linéarité vérifiant une décroissance de « peeling »

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A R T I C L E   I N F O

Article history:
Received 9 January 2010
Accepted after revision 31 May 2010

Presented by Jean-Michel Bony

A B S T R A C T

We prove, outside the influence region of a ball of radius R_0 centered at the origin of the initial data hypersurface, Σ_0, the existence of global solutions near to the Kerr spacetime, provided that the initial data are sufficiently near to those of Kerr. This external region is the "far" part of the outer region of the perturbed Kerr spacetime. Moreover, if we assume that the corrections to the Kerr metric decay sufficiently fast, $o(r^{-3})$, we prove that the various null components of the Riemann tensor decay in agreement with the "Peeling conjecture".

R É S U M É

On démontre, à l'extérieur de la région d'influence d'une boule de rayon R_0 centée à l'origine de l'hypersurface initiale Σ_0, l'existence de solutions globales près de l'espace-temps de Kerr pourvu que les données initiales soient suffisamment proches de celles de Kerr. Cette région extérieure est la partie « éloignée » de la région extérieure de l'espace-temps de Kerr perturbée. De plus si on suppose que les corrections de la métrique de Kerr décroissent suffisamment vite, $o(r^{-3})$, on démontre que la décroissance vers zéro des composantes du tenseur de Riemann est en accord avec la « conjecture de *peeling* ».

Version française abrégée

On démontre la stabilité non linéaire de l'espace-temps de Kerr pour tout moment angulaire J tel que $J \leqslant M^2$ et pour une classe de données initiales au voisinage de Kerr, pour des normes de Sobolev bien choisies, dans une région extérieure où $r \geqslant R_0$ et $M/R_0 \leqslant \lambda$, avec λ suffisamment petit. Le résultat étend de manière significative une démonstration antérieure d'un résultat d'existence donnée dans [5] établi pour une région extérieure suffisamment éloignée de l'horizon des événements de Kerr pour une rotation lente d'un espace-temps de Kerr. De plus on démontre que le comportement asymptotique des composantes de Riemann est compatible avec la conjecture du « *peeling* » si les corrections des données initiales de Kerr décroissent suffisament vite. Ces deux résultats sont contenus dans le Théorème 1.1. On précise que ces deux résultats sont fondamentalement indépendants dans le sens où si on suppose une décroissance plus lente à l'infini en espace

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1631-073X/$ – see front matter
doi:10.1016/j.crma.2010.06.009

(mais pas trop lente) des données initiales on peut démontrer aussi l'existence globale, mais la décroissance asymptotique à l'infini des composantes de Riemann est beaucoup moins bonne que celle suggérée par la conjecture du «*peeling*». On peut aussi estimer assez exactement la perte de décroissance avec la même technique que celle utilisée dans [6] en l'adaptant à la situation actuelle. Ce point est discuté en détail dans l'étape v) de cette Note.

Les principales difficultés techniques rencontrées dans la démonstration de ce résultat sont liées au fait qu'ici la perturbation est non linéaire au voisinage de l'espace temps de Kerr. La différence avec les résultats obtenus dans [3] et [5] est, en gros, que la perturbation est effectuée au voisinage d'une solution non nulle alors que dans les cas précédents on perturbait, dans l'espace-temps de Minkowski, «au voisinage de la solution nulle». Ceci impliquait que toutes les estimations dont nous avions besoin devaient être obtenues en soustrayant des équations utilisées, les équations de transport (ou les équations de structure) dans les coefficients de connexion de la partie associée aux «composantes de Kerr» qui sont, en général, non nulles. Ces points sont discutés dans les étapes i) et ii).

En plus de l'obtention des estimations sur les coefficients de connexion via les équations de transport le long des cônes nuls entrant et sortant, on doit aussi contrôler le tenseur de Riemann le long de ces mêmes cônes. Ainsi, comme pour les coefficients de connexion, on doit soustraire la partie Kerr. Ceci est un point particulièrement délicat que l'on a réussi à faire en remarquant que l'espace-temps de Kerr est stationnaire and que $\frac{\partial}{\partial t}$ est un champ de vecteurs de Killing pour l'espace-temps de Kerr. Ceci suggère que les dérivées de Lie en temps des composantes de Riemann d'un espace-temps de Kerr perturbé ne dépendent pas de la partie du Kerr qui nous était nécessaire. Les estimations *a priori* exigées pour mettre en oeuvre le mécanisme de «*bootstrap*» essentiel dans la démonstration de l'existence doivent donc être obtenues pour des normes L^2 du tenseur de Bel–Robinson (des normes de type énergie dans ce cas) associées à la dérivée de Lie par rapport au temps du champ de vecteurs de Killing de l'espace-temps de Kerr comme on le discute dans les étapes ii), iii) et iv).

1. Introduction

The problem of the global stability for the Kerr spacetime is a very difficult and open problem. The more difficult issue is that of proving the existence of solutions of the vacuum Einstein equations with initial data "near to Kerr" in the whole outer region up to the event horizon.[1]

What is known up to now relative to the whole outer region are some relevant uniform boundedness results for solutions to the wave equation in the Kerr spacetime used as a background spacetime, see Dafermos and Rodnianski [4], and S. Klainerman [7].[2]

If we consider the existence problem in an external region sufficiently far from the Kerr event horizon for a slow rotating Kerr spacetime the result is included in the version of Minkowski stability result proved by S. Klainerman and one of the present authors, see [5] and also [3]. In this case F.N. recently proved, [9], that the asymptotic behaviour of the Riemann components is in agreement with the "Peeling conjecture" if the corrections to the Kerr initial data decay sufficiently fast.

In this note we solve a more difficult problem, namely we prove the nonlinear stability of the Kerr spacetime for any angular momentum J satisfying $J \leqslant M^2$ and an appropriate class of initial data near to Kerr, in an external region where $r \geqslant R_0$ and $M/R_0 \leqslant \lambda$ with λ sufficiently small.

Moreover if we restrict the class of initial data, once subtracted the initial data of the Kerr part, to those which decay toward spacelike infinity faster than r^{-3}, we show again that the null asymptotic decay of the Riemann tensor is in agreement with the "Peeling conjecture". Our main result is summarized in the following theorem[3]:

Theorem 1.1. *Assume that initial data are given on Σ_0 such that, outside of a ball centered in the origin of radius R_0, they are different from the Kerr initial data of a Kerr spacetime with mass M and angular momentum J satisfying $\frac{M}{R_0} \ll 1$, $J \leqslant M^2$, for some metric corrections decaying faster than r^{-3} toward spacelike infinity together with its derivatives up to an order $q \geqslant 4$, namely*[4]

$$g_{ij} = g_{ij}^{(Kerr)} + o_{q+1}\big(r^{-(3+\frac{\gamma}{2})}\big), \qquad k_{ij} = k_{ij}^{(Kerr)} + o_q\big(r^{-(4+\frac{\gamma}{2})}\big) \tag{1}$$

where $\gamma > 0$. Let us also assume that the metric correction, δg_{ij}, and the second fundamental form correction, δk_{ij}, are sufficiently small, namely that we can define a function made by L^2 norms on Σ_0 of these quantities and require it to be small,[5]

$$\mathcal{J}\big(\Sigma_0, R_0; \delta^{(3)}\mathbf{g}, \delta\mathbf{k}\big) \leqslant \varepsilon, \tag{2}$$

then this initial data set has a unique development, $\widetilde{\mathcal{M}}$, defined outside the domain of influence of B_{R_0}. Moreover $\widetilde{\mathcal{M}}$ can be foliated by a double null foliation $\{C(u), \underline{C}(\underline{u})\}$ whose outgoing leaves $C(u)$ are complete[6] *and the various null components of the Riemann tensor relative to a null frame adapted to this foliation decay in agreement with the "Peeling conjecture".*

[1] The event horizon is also an unknown of the problem.
[2] See also for the $J = 0$ case, [1] and references therein.
[3] A more detailed version of the theorem can be found in Section 4 of [2].
[4] The components of the metric tensor written in dimensional coordinates. $f = o_q(r^{-a})$ means that f asymptotically behaves as $o(r^{-a})$ and its partial derivatives $\partial^k f$, up to order q behave as $o(r^{-a-k})$.
[5] Its explicit expression is in [2, Section 3.6, Eq. (3.356)].
[6] By this we mean that the null geodesics generating $C(u)$ can be indefinitely extended toward the future.

The proof of this result depends on many previous results, the result on the stability for the Minkowski spacetime in the external region proved by S. Klainerman and F.N., in [5], a result which, at its turn, is based on the seminal work by D. Christodoulou and S. Klainerman, [3] and, concerning the proof of the peeling decay, important ideas of the proof come from the previous work [6] and the recent paper [9].

We observe that the two results stated in the previous theorem, the global stability in an external region and the asymptotic decay in agreement with the peeling, are basically independent. In the next section we give a sketch of the proof of both results, but it is possible to prove only the first one in a larger class of initial data with weaker decay conditions, see [6] where the analogous result, when perturbing the Minkowski spacetime, has been proved showing how the spacelike decays of the initial data determine the null decays of the Riemann components.

Many technical steps required to prove Theorem 1.1 have been proved in previous works, in the next section we present the central ideas required to prove this result.

2. The proof of Theorem 1.1

To prove the global existence for initial data near the Kerr spacetime in an external region, it is useful first to recall how the problem of perturbing around the Minkowski spacetime solution has been solved in [5]. The general strategy is to apply a "bootstrap argument": one proves, by a local existence result, that there is a finite region V, whose metric satisfies the Einstein equations, endowed with a foliation made by outgoing and incoming null cones, $\{C(u)\}$ and $\{\underline{C}(\underline{u})\}$,[7] and moreover that some $L^p(S)$ or $L^2(C)$ norms for the metric components and their derivatives[8] are bounded by a small constant. One assumes that there exists a largest finite region, V_*, where these "bootstrap bounds" hold and proves that, if the initial data are sufficiently small, these bounds can be improved and the V_* region can be extended. Therefore, to avoid a contradiction, V_* must coincide with the whole (external) spacetime.

To prove that, in V_*, the previous norms have better bounds than those initially assumed one has to look at the structure equations of the spacetime manifold and at its Bianchi equations. The structure equations take the form of transport equations for the connection coefficients along the incoming and outgoing cones and of elliptic Hodge systems on the two dimensional surfaces intersections of the incoming and an outgoing cones, $S = C \cap \underline{C}$. These equations are inhomogeneous and their inhomogeneous terms depend on the Riemann components. The Bianchi equations, in turn, can be written as transport equations for the Riemann components along the cones whose nonlinear part is made by products of Riemann components and connection coefficients.

These equations are used in a sort of linearization argument, we consider the Riemann components as external sources satisfying the "bootstrap bounds", symbolically $|R| \leqslant \epsilon_0$ (and analogously $|O| \leqslant \epsilon_0$ for the connection coefficients),[9] and show, using the equations for connection coefficients, that the bounds of the connection coefficients can be improved, $|O| \leqslant \frac{\epsilon_0}{2}$. Then we control the Riemann tensor using the a priori estimates which can be obtained for the weighted L^2 integrals of the Bel–Robinson tensor, which play the role of the energy norms, associated to the Bianchi equations. These estimates are obtained assuming that the connection coefficients satisfy the "bootstrap bounds" and from them it is possible to prove that these energy-type norms are bounded by the corresponding initial data norms. This allows to obtain, provided initial data sufficiently small, better norms for the Riemann components, $|R| \leqslant \frac{\epsilon_0}{2}$.

The goal of this work is to transport this strategy to perturb nonlinearly around the Kerr spacetime; the difference is, in broad terms, that now we are perturbing around a solution different from zero, while the Minkowski spacetime can be considered a "zero solution". In fact in the Minkowski case all the connection coefficients are identically zero with the exception of the second null fundamental forms χ and $\underline{\chi}$.[10] Moreover all the Riemann components are identically zero. Therefore to "expand around Minkowski" we do not have to subtract the connection coefficients or the Riemann components associated to the Minkowski solution (with the exception of $\operatorname{tr}\chi$ and $\operatorname{tr}\underline{\chi}$). As, viceversa, the Kerr spacetime is not a "zero solution" some kind of subtraction has to be done. This "subtraction" mechanism is central to the proof and delicate as we are not looking to a linearly perturbed solution[11] and it is realized in four different steps:

i) In the first step we define the "bootstrap assumptions" in V_* instead that for the connection coefficients and the Riemann tensor norms for their corrections, that is for the connection coefficients and the Riemann tensor components to which we have subtracted their Kerr parts.[12] Defining, symbolically,

$$\delta O = O - O^{(Kerr)}, \qquad \delta R = R - R^{(Kerr)} \tag{3}$$

and in some more detail, see [5, Chapter 3], for all the definitions,[13]

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[7] Level surfaces of the eikonal equations with appropriate initial conditions. u and $\underline{u}$ are the affine parameters of the null geodesics generating these hypersurfaces.

[8] More precisely relative to the connection coefficients and the components of the Riemann tensor.

[9] The explicit expression of the integral norms for the various components of the Riemann tensor are given in [5, Chapter 3].

[10] Due to the Minkowski spherical symmetry, χ and $\underline{\chi}$ reduce to $\operatorname{tr}\chi = 2r^{-1}$ and $\operatorname{tr}\underline{\chi} = -2r^{-1}$.

[11] Observe that also the linear perturbation around Kerr for the Bianchi equations meets some difficulties due again to the fact that the Riemann tensor in Kerr spacetime is different from zero, see [1] and reference therein.

[12] More precisely the Kerr part "projected" on the V_* foliation associated to the perturbed metric.

[13] α, β are some of the null Riemann tensor components.

$$\delta\chi = \chi - \chi^{(Kerr)}, \qquad \delta\underline{\chi} = \underline{\chi} - \underline{\chi}^{(Kerr)}, \qquad \delta\zeta = \zeta - \zeta^{(Kerr)}, \quad \ldots$$
$$\delta\alpha = \alpha - \alpha^{(Kerr)}, \qquad \delta\beta = \beta - \beta^{(Kerr)}, \quad \ldots$$

the bootstrap assumptions are summarized as

$$|\delta R| \leqslant \epsilon_0, \qquad |\delta O| \leqslant \epsilon_0.$$

ii) In the second step we write the structure equation (in the V_* region) for the connection coefficients to which we have subtracted their Kerr part, $\delta\chi, \delta\underline{\chi}, \delta\zeta, \delta\omega, \delta\underline{\omega}, \ldots$.

These equations are the structure equations for the perturbed Kerr spacetime to which the Kerr part has been subtracted, they have, therefore, inhomogeneous terms which depend on the corrections to the Riemann tensor,

$$\delta R = R - R^{(Kerr)}.$$

We use these modified structure equations to obtain better estimates for the norms of these connection coefficients corrections, $|\delta O| \leqslant \frac{\epsilon_0}{2}$, provided that the norms of the Riemann components corrections, δR satisfy the bootstrap assumptions.

iii) In the third and fourth steps we obtain the improved estimates for the corrections of the Riemann components. Perturbing the Minkowski spacetime the basic step to control the Riemann norms is to prove the boundedness of the energy type norms, $\mathcal{Q}$, L^2 norms of the Bel–Robinson tensor; in the present case, we would need analogous energy-type norms for the Bel–Robinson tensor to which the Kerr part has been subtracted, to get from them estimates for the null Riemann corrections, but we are not able to satisfy this condition in a direct way.[14] Therefore we proceed differently observing that the Kerr spacetime is stationary and $\frac{\partial}{\partial t}$ is a Killing vector field for the Kerr spacetime. This suggests that the time derivatives of the Riemann components of a perturbed Kerr spacetime do not depend on the Kerr part so that their initial data can have a better decay. Therefore if we control the $\mathcal{Q}$ norms of the Bel–Robinson tensor made with the time derivatives of the Riemann tensor we could be able to obtain good estimates for the δR norms in V_* and also a better asymptotic decay along the null directions.

This argument, as presented, is not rigorous, in fact in the perturbed Kerr spacetime $\frac{\partial}{\partial t}$ is not anymore a Killing vector field, but the basic idea can be implemented in the following way: instead of the time derivative of the Riemann tensor we consider its (modified) Lie derivative $\hat{\mathcal{L}}_{T_0} R \equiv \tilde{R}$,[15] where T_0,[16] equal to $\frac{\partial}{\partial t}$ in the Kerr spacetime, is not anymore a Killing vector field, but only "nearly Killing".[17] We define energy-type norms, $\tilde{\mathcal{Q}}$, the analogous of the Q norms defined in [5, Chapter 3], relative to $\tilde{R}$ with appropriate weights and prove, again by a priori estimates, that they are bounded in terms of the corresponding quantities written in terms of the initial data. Their first terms, see for details [2], have the following expressions,

$$\begin{aligned}
&\widetilde{\mathcal{Q}}(u,\underline{u}) = \widetilde{\mathcal{Q}}_1(u,\underline{u}) + \widetilde{\mathcal{Q}}_2(u,\underline{u}) + \sum_{i=2}^{q-1} \widetilde{\mathcal{Q}}_{(q)}(u,\underline{u}),\\
&\underline{\widetilde{\mathcal{Q}}}(u,\underline{u}) = \underline{\widetilde{\mathcal{Q}}}_1(u,\underline{u}) + \underline{\widetilde{\mathcal{Q}}}_2(u,\underline{u}) + \sum_{i=2}^{q-1} \underline{\widetilde{\mathcal{Q}}}_{(q)}(u,\underline{u}),\\
&V(u,\underline{u}) = J^{(-)}\big(S(u,\underline{u})\big) \cap V_*,
\end{aligned} \tag{4}$$

$$\begin{aligned}
\widetilde{\mathcal{Q}}_1(u,\underline{u}) \equiv& \int_{C(u)\cap V(u,\underline{u})} |u|^{5+\gamma} Q(\hat{\mathcal{L}}_T \tilde{R})(\bar{K},\bar{K},\bar{K},e_4) + \int_{C(u)\cap V(u,\underline{u})} |u|^{5+\gamma} Q(\hat{\mathcal{L}}_O \tilde{R})(\bar{K},\bar{K},T,e_4),\\
\widetilde{\mathcal{Q}}_2(u,\underline{u}) \equiv& \int_{C(u)\cap V(u,\underline{u})} |u|^{5+\gamma} Q(\hat{\mathcal{L}}_O \hat{\mathcal{L}}_T \tilde{R})(\bar{K},\bar{K},\bar{K},e_4) + \int_{C(u)\cap V(u,\underline{u})} |u|^{5+\gamma} Q(\hat{\mathcal{L}}_O^2 \tilde{R})(\bar{K},\bar{K},T,e_4)\\
&+ \int_{C(u)\cap V(u,\underline{u})} |u|^{5+\gamma} Q(\hat{\mathcal{L}}_S \hat{\mathcal{L}}_T \tilde{R})(\bar{K},\bar{K},\bar{K},e_4)
\end{aligned} \tag{5}$$

where, denoting with the * the Hodge dual,

$$Q(W)(X,Y,Z,e_4) = Q(W)_{\mu\nu\rho\sigma} X^\mu Y^\nu Z^\rho e_4^\sigma, \qquad Q(W)_{\mu\nu\rho\sigma} = W_{\mu\tau\rho\delta} W_\nu{}^\tau{}_\sigma{}^\delta + {}^*W_{\mu\tau\rho\delta}\, {}^*W_\nu{}^\tau{}_\sigma{}^\delta \tag{6}$$

[14] Trying to subtract the Kerr part to the Bel–Robinson tensor conflicts with the need of a positive definite integrand for the $\delta\mathcal{Q}$ norms, see the explicit expression of the $\mathcal{Q}$ norms in Minkowski case in [5, Chapter 3].

[15] See [5, Chapter 3], for its precise definition.

[16] Its precise definition is in [2].

[17] With "nearly Killing" we mean that its deformation tensor is small with respect to some Sobolev norms.

and $\gamma > 0$. The remaining terms are those relative to incoming cones and terms depending on higher order Lie derivatives. These energy-type estimates, some times called "flux integrals", allow to control the components of the $\tilde{R}$ tensor along the incoming and outgoing cones inside V_*. Their weights are due to the presence of the Morawetz type $\bar{K}$ vector fields

$$\bar{K} = \frac{1}{2}\left(\sqrt{1+\underline{u}^2}e_4 + \sqrt{1+u^2}e_3\right),$$

but also to the term $|u|^{5+\gamma}$. This last factor is the one needed to prove that the Riemann components decay in agreement with the "Peeling conjecture".[18]

iv) In the fourth step we follow the strategy developed when perturbing the Minkowski spacetime; from the a priori estimates for the $\tilde{Q}$ norms, which require, to be proved, M/R_0 sufficiently small, we estimate these norms in terms of the initial data ones. Using the bootstrap assumptions for the $|\delta O|$ norms and the Bianchi equations we control the $\tilde{R}$ norms in terms of the initial data. After quite a few steps, assuming the "bootstrap bounds" for the δO norms and that the initial data are sufficiently small (symbolically $\leqslant \varepsilon$ with $\varepsilon \ll \epsilon_0$), we control the time derivatives of the various δR norms and with a time integration we prove that the δR norms are smaller than originally assumed, $|\delta R| \leqslant \frac{\epsilon_0}{2}$, and satisfy appropriate decays. This completes the main step required to prove (external) global existence.[19]

v) The decay of the Riemann components, the peeling:

The previous steps sketch the proof of the Kerr stability in a region with $r \geqslant R_0 \gg M$, but Theorem 1.1 proves also that the null Riemann components have a null asymptotic decay consistent with the "Peeling conjecture".[20] In fact, once we have proved a global existence result, the way to prove again the "peeling decay" in this larger region is similar to the one discussed in [9]. We describe the main ideas involved and refer to [9] and [2] for a more detailed discussion.

In [3] and in [5], the asymptotic behaviour along the null outgoing directions of some of the Riemann tensor components, specifically the α and the β components, is slower than the one expected from the "Peeling conjecture", [10]. More precisely the components α and β[21] decay as $r^{-\frac{7}{2}}$ while from the "Peeling conjecture" one expects r^{-5} and r^{-4} respectively.[22] In a subsequent paper, [6], S. Klainerman and one of us proved that the expected decay could be obtained assuming stronger decay for the initial data. Unfortunately the required initial data were incompatible with the Kerr initial data. In fact the needed decay was (here with g_{ij} we mean the components written in Cartesian coordinates),

$$g_{ij} = g_{S\,ij} + O_{q+1}\left(r^{-(3+\epsilon)}\right), \qquad k_{ij} = O_q\left(r^{-(4+\epsilon)}\right) \tag{7}$$

where g_S denotes the restriction of the Schwarzschild metric on the initial hypersurface, and it is immediate to see that Kerr spacetime gives terms, beside the Schwarzschild part, decaying as $O(r^{-2})$. To prove the result it is crucial that, again, the Kerr part be subtracted. In fact, as explained in a moment, to obtain the "Peeling decay" we add the $|u|^{5+\gamma}$ weight in the "flux integrals and, on the initial hypersurface, $|u|$ behaves like r producing divergent initial energy-type norms if we do not subtract the Kerr part.

Proceeding as described in the previous steps we control first the $\tilde{R}$ norms with appropriate $|u|$ weights, more precisely we prove the following bounds, with $N_0 \gg 1$,[23]

$$\sup_{V_*} r^{\frac{7}{2}}|u|^{\frac{5}{2}+\frac{\epsilon}{2}}\left|\alpha(\hat{\mathcal{L}}_{T_0}R)\right| \leqslant c_2\left(\varepsilon + \frac{\epsilon_0}{N_0}\right), \qquad \sup_{V_*} r^{\frac{7}{2}}|u|^{\frac{5}{2}+\frac{\epsilon}{2}}\left|\beta(\hat{\mathcal{L}}_{T_0}R)\right| \leqslant c_2\left(\varepsilon + \frac{\epsilon_0}{N_0}\right). \tag{8}$$

Next crucial step consists in proving that, integrating along the incoming cones, the extra decay in the $|u|$ variable can be transformed in an extra decay in the r variable, as was first proved in [6] for the Minkowski perturbed spacetimes. This allows to get the following estimates

$$\sup_{V_*} r^5|u|^{1+\frac{\epsilon}{2}}\left|\alpha(\hat{\mathcal{L}}_{T_0}R)\right| \leqslant \tilde{c}_4\left(\varepsilon + \frac{\epsilon_0}{N_0}\right), \qquad \sup_{V_*} r^4|u|^{2+\frac{\epsilon}{2}}\left|\beta(\hat{\mathcal{L}}_{T_0}R)\right| \leqslant \tilde{c}_3\left(\varepsilon + \frac{\epsilon_0}{N_0}\right). \tag{9}$$

Finally as we said in step iv) we recover the control of the δR norms with a time integration which is convergent due to the $|u|^{1+\frac{\epsilon}{2}}$ and $|u|^{2+\frac{\epsilon}{2}}$ factors in the previous estimates which, at fixed r, behave as $|t|$ factors. In this way together with the global existence in the external region we have that δR satisfies the "Peeling decay". As also the Kerr part of the Riemann tensor satisfies it, the result is achieved.

Summarizing the previous discussion we can say that the result of this Note is that we have been able to define an external region where the global solution of a Kerr perturbation exists and where the "Peeling conjecture" is satisfied. This has been possible mainly due to the delicate subtraction of the "Kerr part".

[18] Its introduction is not harmful as in controlling the $\tilde{\mathcal{Q}}$ norms in terms of the initial data ones, the error to be estimated is modified by an extra term with the right sign.

[19] How the global existence is a consequence of this result follows the line of arguments discussed in [5].

[20] This result was already obtained in [9], restricting to the perturbation of a very slow rotating Kerr spacetime or to a "very external region".

[21] Components relative to a null frame adapted to the null outgoing and incoming cones which foliate the "external region".

[22] In principle some log powers can be present, see J.A.V. Kroon, [8].

[23] The analogous ones for the remaining components are less important as these terms already satisfy the "Peeling decay".

Finally it has to be pointed out, that, even if we improve the kind of estimates we are using here, this strategy does not allow us to cover the whole outer region, (that is from the event horizon on). This should be evident already looking at [1] where it is clear that the control of the "error", crucial to control the "energy-norms", for the linear wave equation in Schwarzschild background spacetime, is very problematic, around the so called "photosphere region". This is one of the basic reasons why the assumption $\frac{M}{R_0} \ll 1$ is required in Theorem 1.1 and it is obvious that more refined technical tools and new ideas are required to cope with this region in the Einstein equations case.

Acknowledgements

One of the authors, F.N. is deeply indebted to S. Klainerman for pointing to him the importance of considering the Lie derivative with respect to the "time" vector field T_0, of the Riemann tensor to obtain more detailed estimates for the various components of the Riemann tensor. Beside F.N. is also indebted for many illuminating discussions he had with him about this subject and many related ones. We also want to state clearly that the present result is deeply based on the previous works [5,6] and on the original fundamental work by D. Christodoulou and S. Klainerman [3].

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