C. R. Acad. Sci. Paris, Ser. I 348 (2010) 843–846

Contents lists available at ScienceDirect

C. R. Acad. Sci. Paris, Ser. I

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Mathematical Analysis

Hyperbolicity preservers and majorization

Préservateurs d'hyperbolicité et majorisation

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ARTICLE INFO

Article history:
Received 7 June 2010
Accepted after revision 8 July 2010
Available online 29 July 2010

Presented by Jean-Pierre Kahane

ABSTRACT

The majorization order on $\mathbb{R}^n$ induces a natural partial ordering on the space of univariate hyperbolic polynomials of degree n. We characterize all linear operators on polynomials that preserve majorization, and show that it is sufficient (modulo obvious degree constraints) to preserve hyperbolicity.

RÉSUMÉ

L'ordre de majorisation de $\mathbb{R}^n$ induit un ordre partiel naturel sur l'espace des polynômes hyperboliques univariés de degré n. Nous caractérisons les opérateurs linéaires sur ces polynômes préservant l'ordre donné et montrons que seule la préservation de l'hyperbolicité suffit (modulo des contraintes évidentes sur le degré).

1. Introduction and main result

A polynomial in $\mathbb{R}[z]$ is *hyperbolic* if it has only real zeros. The space $\mathcal{H}_n$ of all hyperbolic polynomials of degree n is equipped with a natural partial ordering defined in terms of the majorization order on weakly increasing vectors in $\mathbb{R}^n$. If $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_n)$ are weakly increasing vectors in $\mathbb{R}^n$, then y *majorizes* x (denoted $x \prec y$) if $\sum_{i=1}^n x_i = \sum_{i=1}^n y_i$, and $\sum_{i=0}^k x_{n-i} \leqslant \sum_{i=0}^k y_{n-i}$ for each $0 \leqslant k \leqslant n-2$. Given a polynomial $p \in \mathcal{H}_n$ arrange the zeros (counting multiplicities) of p in a weakly increasing vector $\mathcal{Z}(p) \in \mathbb{R}^n$. If $p, q \in \mathcal{H}_n$ we say that p *is majorized by* q, denoted $p \prec q$, if p and q have the same leading coefficient and $\mathcal{Z}(p) \prec \mathcal{Z}(q)$. In particular if $p \prec q$, then the top two coefficients of p and q are the same. The majorization order on $\mathcal{H}_n$ was studied in [1,2,4,6,12]. Particular interest has been given to the question of which linear operators on polynomials preserve majorization. The purpose of this note is to characterize such operators.

Let $\mathbb{R}_n[z]$ be the linear space of all real polynomials of degree at most n. A linear operator $T : \mathbb{R}_n[z] \to \mathbb{R}[z]$ *preserves majorization* if $T(p) \prec T(q)$ whenever $p, q \in \mathcal{H}_n$ are such that $p \prec q$. Recall that two hyperbolic polynomials have *interlacing zeros* if

$$x_1 \leqslant y_1 \leqslant x_2 \leqslant y_2 \leqslant \cdots \quad \text{or} \quad y_1 \leqslant x_1 \leqslant y_2 \leqslant x_2 \leqslant \cdots,$$

where $x_1 \leqslant x_2 \leqslant \cdots$ and $y_1 \leqslant y_2 \leqslant \cdots$ are the zeros of p and q, respectively. We say that a polynomial $p(z_1, \dots, z_n) \in \mathbb{C}[z_1, \dots, z_n]$ is *stable* if it is nonzero whenever all variables have positive imaginary parts. A linear operator $T : \mathbb{R}_n[z] \to \mathbb{R}[z]$

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[1] The author is a Royal Swedish Academy of Sciences Research Fellow supported by a grant from the Knut and Alice Wallenberg Foundation.

1631-073X/$ – see front matter
doi:10.1016/j.crma.2010.07.006

is called *degenerate* if $\dim(T(\mathbb{R}_n[z])) \leqslant 2$. The *symbol* of a linear operator $T : \mathbb{R}_n[z] \to \mathbb{R}[z]$ is the bivariate polynomial $F_T(z,w) = \sum_{k=0}^{n} \binom{n}{k} T(z^k) w^{n-k}$. The following theorem is our main result and will be proved in the next section:

Theorem 1. *Suppose that $T : \mathbb{R}_n[z] \to \mathbb{R}[z]$ is a linear operator, where $n \geqslant 1$. Then T preserves majorization if and only if*

(1) *T is nondegenerate and $T(\mathcal{H}_n) \subseteq \mathcal{H}_m$ for some m, or*
(2) *T is of the form $T(\sum_{k=0}^{n} a_k z^k) = a_n T(z^n) + a_{n-1} T(z^{n-1})$, where $T(z^n) \not\equiv 0$ is hyperbolic, and either $T(z^{n-1}) \equiv 0$ or $T(z^{n-1})$ is a hyperbolic polynomial which is not a constant multiple of $T(z^n)$, and $T(z^{n-1})$ and $T(z^n)$ have interlacing zeros.*

Moreover, condition (1) is equivalent to that T is nondegenerate, and $F_T(z,w)$ or $F_T(z,-w)$ is stable and such that $\deg(T(z^n)) > \deg(T(z^k))$ for all $k < n$.

Theorem 1 complements [3] where the authors characterized all linear operators on polynomials preserving hyperbolicity. Also, Theorem 1 answers in the affirmative several questions raised in [1,2].

2. Proof of Theorem 1

We will use the algebraic characterization of hyperbolicity preservers obtained in [3]:

Theorem 2. *Suppose that $T : \mathbb{R}_n[z] \to \mathbb{R}[z]$ is a linear operator, where $n \geqslant 1$. Then T preserves hyperbolicity if and only if*

- *T is degenerate and is of the form*

$$T(p) = \alpha(p)P + \beta(p)Q,$$

where $\alpha, \beta : \mathbb{R}_n[z] \to \mathbb{R}$ are linear functionals and P, Q are hyperbolic polynomials with interlacing zeros, or
- *T is nondegenerate and $F_T(z,w)$ is stable, or*
- *T is nondegenerate and $F_T(z,-w)$ is stable.*

Suppose first that T is degenerate. If T is as in (2) of Theorem 1, then T preserves hyperbolicity by Obreshkoff's theorem, see e.g. [3, Theorem 10]. Also, $T(p) = T(q)$ whenever $p \prec q$ which proves that (2) is sufficient. Note that if $p = \sum_{k=0}^{n} a_k z^k \in \mathcal{H}_n$, then $a_n(z + a_{n-1}/na_n)^n \prec p$. Hence if T preserves majorization, then the degree and the top two coefficients of $T(f)$ only depend on the top two coefficients of p. Since T is of the form $T(p) = \alpha(p)P + \beta(p)Q$, where α and β are functionals (by Theorem 2) it is not hard to see that T has to be of the form (2). Henceforth, we assume that T is nondegenerate. We start by proving that (1) is sufficient.

Lemma 3. *Suppose that $T : \mathbb{R}_n[z] \to \mathbb{R}[z]$ is a nondegenerate linear operator preserving hyperbolicity. Then there are numbers $0 \leqslant K \leqslant L \leqslant M \leqslant N \leqslant n$ such that*

(1) *$T(z^k) \equiv 0$ if $k < K$ or $k > N$;*
(2) *$\deg(T(z^{k+1})) = \deg(T(z^k)) + 1$ for all $K \leqslant k < L$;*
(3) *$\deg(T(z^k)) \leqslant \deg(T(z^L)) = \deg(T(z^M))$ for all $L \leqslant k \leqslant M$, and*
(4) *$\deg(T(z^{k+1})) = \deg(T(z^k)) - 1$ for all $M \leqslant k < N$.*

Proof. By Theorem 2, either $F_T(z,w)$ or $F_T(z,-w)$ is stable. The lemma is a simple consequence of the fact that the support of a stable polynomial is a jump system, see [5, Theorem 3.2]. □

Remark 1. Suppose that $T : \mathbb{R}_n[z] \to \mathbb{R}[z]$ is a nondegenerate linear operator such that $T(\mathcal{H}_n) \subseteq \mathcal{H}_m$. Since any hyperbolic polynomial of degree at most n is the limit of degree n polynomials, it follows from Hurwitz' theorem on the continuity of zeros that T preserves hyperbolicity. But then $L = M = N = n$, since otherwise one could produce two polynomials $p, q \in \mathcal{H}_n$ such that $\deg(T(p)) \neq \deg(T(q))$.

To any nondegenerate hyperbolicity preserver, we associate a sequence $\{\gamma_k(T)\}_{k=0}^{n}$ by defining $\gamma_k(T)$ to be the coefficient in front of z^{r+k} in $T(z^k)$, where $r = \deg(T(z^K)) - K$ and K is as in Lemma 3. We claim that the linear operator $\Gamma : \mathbb{R}_n[z] \to \mathbb{R}[z]$ defined by $\Gamma(z^k) = \gamma_k(T) z^k$ preserves hyperbolicity. Indeed,

$$\Gamma\big(p(z)\big) = \lim_{\rho \to 0} (\rho/z)^r T\big(p(\rho z)\big)(z/\rho),$$

so the claim follows from Hurwitz' theorem.

Remark 2. It is known that such sequences have either constant sign or are alternating in sign, and that the indices k for which $\gamma_k(T) \neq 0$ form an interval, see e.g. [7, Theorem 3.4].

To prove Theorem 1 we will use an important result on hyperbolic polynomials in several variables. A homogeneous polynomial $p \in \mathbb{R}[z_1, \dots, z_n]$ is said to be *hyperbolic* with respect to a vector $e \in \mathbb{R}^n$ if $p(e) \neq 0$ and for all vectors $\alpha \in \mathbb{R}^n$ the polynomial $p(\alpha + et) \in \mathbb{R}[t]$ has only real zeros. The following theorem, proved by Lewis, Parrilo and Ramana based heavily on the work of Dubrovin, Helton–Vinnikov and Vinnikov, settled the so-called Lax conjecture.

Theorem 4. *(See [9,10].) Let $p \in \mathbb{R}[x, y, z]$ be a homogeneous polynomial of degree m. Then p is hyperbolic with respect to $e = (1, 0, 0)$ if and only if there exist two real symmetric $m \times m$ matrices B and C such that*

$$p(x, y, z) = p(e)\det(xI - yB - zC).$$

Theorem 4 enables us to use the following well-known convexity result in matrix theory due to K. Fan.

Lemma 5. *(See [8].) Let A be a complex Hermitian matrix of size $n \times n$, and denote by $\lambda_1(A) \leqslant \dots \leqslant \lambda_n(A)$ its eigenvalues arranged in weakly increasing order. For each $1 \leqslant k \leqslant n$ the function*

$$A \mapsto \sum_{i=1}^{k} \lambda_{n+1-i}(A)$$

is convex on the real space of Hermitian $n \times n$ matrices.

Lemma 6. *Let $T : \mathbb{R}_n[z] \to \mathbb{R}[z]$ be a nondegenerate linear operator satisfying $T(\mathcal{H}_n) \subseteq \mathcal{H}_m$, where $n \geqslant 2$. Let further $r(z) \in \mathcal{H}_{n-2}$ be monic, and s be a fixed real number. For $t \in \mathbb{R}$, let $x_1(t) \leqslant \dots \leqslant x_m(t)$ be the zeros of the polynomial $T(r(z)((z+s)^2 - t^2))$. Then for each $1 \leqslant k \leqslant m$,*

$$\mathbb{R} \ni t \mapsto \sum_{i=1}^{k} x_{m+1-i}(t) \tag{1}$$

is a convex and even function on $\mathbb{R}$. Moreover,

$$T\big(r(z)\big((z+s)^2 - t_1^2\big)\big) \prec T\big(r(z)\big((z+s)^2 - t_2^2\big)\big),$$

whenever $0 \leqslant t_1 \leqslant t_2$.

Proof. Set $g(z) = T(r(z)(z+s)^2)$, $h(z) = T(r(z))$, and $m = \deg g$. If $h \equiv 0$ there is nothing to prove so we may assume that $\deg h \geqslant 0$. Then $\deg h = m - 2$ by Remark 1. We claim that the homogeneous degree m polynomial in three variables

$$f(z_1, z_2, z_3) = z_3^m g(z_1/z_3) - z_2^2 z_3^{m-2} h(z_1/z_3)$$

is hyperbolic with respect to the vector $e = (1, 0, 0)$. If $\alpha = (a, b, 0)$, then

$$f(\alpha + et) = \gamma_m(T)(a+t)^m - b^2\gamma_{m-2}(T)(a+t)^{m-2}$$

has only real zeros since, by Remark 2, $\gamma_m(T)\gamma_{m-2}(T) > 0$. Also, if $\alpha = (a, b, c)$ where $c \neq 0$, then

$$f(\alpha + et) = c^m T\big(r(z)\big(z^2 - b^2/c^2\big)\big)\big|_{z=(a+t)/c}$$

has only real zeros, and the claim follows.

By Theorem 4 there exist real symmetric $m \times m$ matrices B and C such that

$$f(z_1, z_2, z_3) = f(e)\det(z_1 I - z_2 B - z_3 C).$$

It follows that for any fixed $t \in \mathbb{R}$ the zeros of the polynomial

$$T\big(r(z)\big((z+s)^2 - t^2\big)\big) = f(z, t, 1) = g(z) - t^2 h(z)$$

are precisely the eigenvalues of the real symmetric matrix $tB + C$. Note also that $\sum_{i=1}^{m} x_i(t)$ is constant in t, since the two top coefficients of $f(z, t, 1)$ come from $g(z)$. The lemma now follows from Lemma 5. □

To complete the proof of the sufficiency of (1) in Theorem 1 we need a well-known lemma due to Hardy, Littlewood and Pólya, see [11]. For simplicity, we formulate it by means of polynomials in $\mathcal{H}_n$. Given $p, q \in \mathcal{H}_n$ with $n \geqslant 2$, $\mathcal{Z}(p) = (x_1, \dots, x_n)$ and $\mathcal{Z}(q) = (y_1, \dots, y_n)$ we say that p is a *pinch* of q if there exist $1 \leqslant i \leqslant n-1$ and $0 \leqslant t \leqslant (y_{i+1} - y_i)/2$ such that $x_i = y_i + t$, $x_{i+1} = y_{i+1} - t$, and $x_k = y_k$ for $k \neq i$. Note that if p is a pinch of q, then we may write p and q as $p(z) = r(z)((z+s)^2 - t_1^2)$ and $q(z) = r(z)((z+s)^2 - t_2^2)$, where r is a hyperbolic polynomial and $s, t_1, t_2 \in \mathbb{R}$ with $0 \leqslant t_1 \leqslant t_2$.

Lemma 7. *If $p, q \in \mathcal{H}_n$, $n \geqslant 2$, are such that $p \prec q$, then p may be obtained from q by a finite number of pinches.*

Suppose now that $p \prec q \in \mathcal{H}_n$ where $n \geqslant 2$ and that T is as in (1) of Theorem 1. By Lemma 7 there are polynomials $p = p_0, p_1, \ldots, p_k = q$ in $\mathcal{H}_n$ such that p_{i-1} is a pinch of p_i for all $1 \leqslant i \leqslant k$. By Lemma 6, $T(p_{i-1}) \prec T(p_i)$ for all $1 \leqslant i \leqslant k$ so by transitivity $T(p) \prec T(q)$. The case when $n = 1$ follows from the case when $n = 2$ by considering the map T' defined by $T'(f) = T(f')$.

To prove the remaining direction in Theorem 1 assume that T preserves majorization. If $\deg(T(z^n)) > \deg(T(z^{n-1}))$, then by Lemma 3, $\deg(T(p)) = \deg(T(q))$ for any two polynomials p, q of degree n. In particular $T(\mathcal{H}_n) \subseteq \mathcal{H}_m$ for some m. Assume that $\deg(T(z^n)) \leqslant \deg(T(z^{n-1}))$. Recall that $\deg(T(p))$ and the top two coefficients of $T(p)$ only depend on the top two coefficients of p. This can only happen if $\deg(T(z^{n-2})) \leqslant \deg(T(z^{n-1})) - 2$, since otherwise the top two coefficients of $T(z^n - a^2 z^{n-2})$ would depend on the real parameter a. But then, by Lemma 3, $T(1) \equiv \cdots \equiv T(z^{n-2}) \equiv 0$ and T is thus degenerate contrary to the assumptions.

The final sentence in Theorem 1 follows from Lemma 3 and Theorem 2.

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