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Partial Differential Equations

Global well-posedness theory for the spatially inhomogeneous Boltzmann equation without angular cutoff

Existence globale pour l'équation de Boltzmann sans troncature

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ABSTRACT

We present the first global well-posedness result for the Boltzmann equation without angular cutoff in the framework of weighted Sobolev spaces, in a close to equilibrium framework, and for Maxwellian molecules. These solutions become smooth for any positive time. An important ingredient of the proof rests on the introduction of a new norm, encoding both the singularity and the dissipation properties of the linearized collision operator.

RÉSUMÉ

Nous présentons le premier résultat d'existence globale pour l'équation de Boltzmann sans troncature angulaire, dans le cadre des espaces de Sobolev à poids, dans un cadre proche de l'équilibre, et pour des molécules maxwelliennes. Ces solutions devienent régulières pour tout temps positif. Un point important de la preuve consiste en l'introduction d'une nouvelle norme adaptée à la singularité et aux propriétés de dissipation de l'opérateur de collision linéarisé.

Version française abrégée

Soit l'équation de Boltzmann $f_t + v \cdot \nabla_x f = Q(f, f)$, où $f = f(t, x, v)$ est la densité de particules, avec la position $x \in \mathbb{R}^3$ et la vitesse $v \in \mathbb{R}^3$ au temps t. L'opérateur de Boltzmann est donné par

$$Q(g, f) = \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} B(v - v_*, \sigma) \{g'_* f' - g_* f\} \, \mathrm{d}\sigma \, \mathrm{d}v_*,$$

où $f'_* = f(t, x, v'_*), \ldots,$ et pour $\sigma \in \mathbb{S}^2$, les vitesses post et pré collisionnelles sont données par $v' = \frac{v+v_*}{2} + \frac{|v-v_*|}{2}\sigma$ et $v'_* = \frac{v+v_*}{2} - \frac{|v-v_*|}{2}\sigma$.

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On suppose que le noyau B est du type maxwellien

$$B\big(|v-v_*|,\cos\theta\big)=b(\cos\theta),\quad \cos\theta=\left\langle\frac{v-v_*}{|v-v_*|},\sigma\right\rangle,\ 0\leqslant\theta\leqslant\frac{\pi}{2},\tag{1}$$

et

$$b(\cos\theta)\approx K\theta^{-2-2s},\quad \theta\to 0^+,\ 0<s<1/2.\tag{2}$$

En posant $\mu(v)=(2\pi)^{-\frac{3}{2}}e^{-\frac{|v|^2}{2}}$, $f=\mu+\sqrt{\mu}g$, $\Gamma(g,h)=\mu^{-1/2}Q(\sqrt{\mu}g,\sqrt{\mu}h)$, et $\mathcal{L}g=\mathcal{L}_1g+\mathcal{L}_2g\equiv-\Gamma(\sqrt{\mu},g)-\Gamma(g,\sqrt{\mu})$, le problème originel est réduit au problème de Cauchy suivant

$$\begin{cases} g_t+v\cdot\nabla_x g+\mathcal{L}g=\Gamma(g,g),\quad t>0;\\ g|_{t=0}=g_0.\end{cases}\tag{3}$$

Pour $k,\ell\in\mathbb{R}$, on introduit $H^k_\ell(\mathbb{R}^6_{x,v})=\{f\in\mathcal{S}'(\mathbb{R}^6_{x,v});\ W^\ell f\in H^k(\mathbb{R}^6_{x,v})\}$, avec $W^\ell(v)=\langle v\rangle^\ell=(1+|v|^2)^{\ell/2}$.
Le résultat principal de cette Note est donné par le théorème suivant :

Théorème 0.1. *Sous les hypothèses* (1), (2), *soit* $g_0\in H^k_\ell(\mathbb{R}^6)$ *avec* $k\geqslant 3$, $\ell\geqslant 3$ *et* $f_0(x,v)=\mu+\sqrt{\mu}g_0(x,v)\geqslant 0$. *Alors il existe* $\varepsilon_0>0$, *tel que si* $\|g_0\|_{H^k_\ell(\mathbb{R}^6)}\leqslant\varepsilon_0$, *le problème de Cauchy* (3) *admet une unique solution globale* $g\in L^\infty([0,+\infty[;H^k_\ell(\mathbb{R}^6))$ *avec* $f(t,x,v)=\mu+\sqrt{\mu}g(t,x,v)\geqslant 0$ *et* $g\in C^\infty(]0,+\infty[\times\mathbb{R}^6)$.

On pourra trouver plus de détails dans la version anglaise, les preuves complètes faisant l'objet d'une prépublication [4].

1. Introduction

Consider the Boltzmann equation $f_t+v\cdot\nabla_x f=Q(f,f)$, where $f=f(t,x,v)$ is the distribution density of particles, with position $x\in\mathbb{R}^3$ and velocity $v\in\mathbb{R}^3$ at time t. The right hand side is given by the Boltzmann bilinear collision operator

$$Q(g,f)=\int_{\mathbb{R}^3}\int_{\mathbb{S}^2}B(v-v_*,\sigma)\big\{g'_*f'-g_*f\big\}\,d\sigma\,dv_*,$$

where $f'_*=f(t,x,v'_*),\dots$, and for $\sigma\in\mathbb{S}^2$, the post and pre collisional velocities are given by $v'=\frac{v+v_*}{2}+\frac{|v-v_*|}{2}\sigma$ and $v'_*=\frac{v+v_*}{2}-\frac{|v-v_*|}{2}\sigma$. We assume that B behaves like a non cut-off Maxwellian type cross section:

$$B\big(|v-v_*|,\cos\theta\big)=b(\cos\theta),\quad \cos\theta=\left\langle\frac{v-v_*}{|v-v_*|},\sigma\right\rangle,\ 0\leqslant\theta\leqslant\frac{\pi}{2},\tag{4}$$

and

$$b(\cos\theta)\approx K\theta^{-2-2s},\quad \theta\to 0^+,\ 0<s<1/2.\tag{5}$$

Similarly to the cut-off studies, see for instance [5], setting $\mu(v)=(2\pi)^{-\frac{3}{2}}e^{-\frac{|v|^2}{2}}$, letting $f=\mu+\sqrt{\mu}g$, introducing $\Gamma(g,h)=\mu^{-1/2}Q(\sqrt{\mu}g,\sqrt{\mu}h)$, and

$$\mathcal{L}g=\mathcal{L}_1g+\mathcal{L}_2g\equiv-\Gamma(\sqrt{\mu},g)-\Gamma(g,\sqrt{\mu}),$$

we will study the following Cauchy problem

$$\begin{cases} g_t+v\cdot\nabla_x g+\mathcal{L}g=\Gamma(g,g),\quad t>0,\\ g|_{t=0}=g_0.\end{cases}\tag{6}$$

Let $H^k_\ell(\mathbb{R}^6_{x,v})=\{f\in\mathcal{S}'(\mathbb{R}^6_{x,v});\ W^\ell f\in H^k(\mathbb{R}^6_{x,v})\}$ be the weighted Sobolev space, $k,\ell\in\mathbb{R}$ and $W^\ell(v)=\langle v\rangle^\ell=(1+|v|^2)^{\ell/2}$.
The main result of this Note is given by the following:

Theorem 1.1. *Assuming* (4), (5), *with* $0<s<1/2$, *let* $g_0\in H^k_\ell(\mathbb{R}^6)$ *for some* $k\geqslant 3$, $\ell\geqslant 3$ *and* $f_0(x,v)=\mu+\sqrt{\mu}g_0(x,v)\geqslant 0$. *There exists* $\varepsilon_0>0$, *such that if* $\|g_0\|_{H^k_\ell(\mathbb{R}^6)}\leqslant\varepsilon_0$, *the Cauchy problem* (6) *admits a unique global solution* $g\in L^\infty([0,+\infty[;H^k_\ell(\mathbb{R}^6))$ *with* $f(t,x,v)=\mu+\sqrt{\mu}g(t,x,v)\geqslant 0$ *and* $g\in C^\infty(]0,+\infty[\times\mathbb{R}^6)$.

The details of the proof will appear in the full version, [4], and here we only sketch the main steps.

2. Main arguments

2.1. Functional properties of the collision operator

It was proven in [1] the following coercivity estimate:

$$c_g\|f\|^2_{H^s(\mathbb{R}^3_v)} \leqslant \big(-Q(g,f),f\big)_{L^2(\mathbb{R}^3_v)} + C\|g\|_{L^1(\mathbb{R}^3_v)}\|f\|^2_{L^2(\mathbb{R}^3_v)}, \tag{7}$$

for any $g \geqslant 0$, $g \in L^1_2 \cap L\log L(\mathbb{R}^3_v)$ and $f \in H^s(\mathbb{R}^3_v)$, where $c_g \geqslant 0$ depending only on $\|g\|_{L^1_2}$ and $\|g\|_{L\log L}$. And in [3] the following upper bound estimate was proven: for any $m, \alpha \in \mathbb{R}$, there exists $C > 0$ such that

$$\big\|Q(f,g)\big\|_{H^m_\alpha(\mathbb{R}^3_v)} \leqslant C\|f\|_{L^1_{\alpha^+ + 2s}(\mathbb{R}^3_v)}\|g\|_{H^{m+2s}_{(\alpha+2s)^+}(\mathbb{R}^3_v)}. \tag{8}$$

For the linearized operators, it is well known that $\mathcal{L}$ is an unbounded symmetric positive operator on $L^2(\mathbb{R}^3_v)$, and that $(\mathcal{L}g,g)_{L^2(\mathbb{R}^3_v)} = 0$ iff $\mathbf{P}g = g$, where $\mathbf{P}g = (a + b\cdot v + c|v|^2)\sqrt{\mu}$, with $a, c \in \mathbb{R}$, $b \in \mathbb{R}^3$, $\mathbf{P}$ being the L^2-orthogonal projection onto the null space $\mathrm{Ker}(\mathcal{L}) = \mathrm{Span}\{\sqrt{\mu}, v_1\sqrt{\mu}, v_2\sqrt{\mu}, v_3\sqrt{\mu}, |v|^2\sqrt{\mu}\}$. From [6], there exists a constant $C > 0$ such that $(\mathcal{L}g,g)_{L^2(\mathbb{R}^3_v)} \geqslant C\|(g - \mathbf{P}g)\|^2_{L^2_s(\mathbb{R}^3_v)}$. Furthermore, $(\mathcal{L}_1 g,g)_{L^2(\mathbb{R}^3_v)} \geqslant \frac{1}{2}(\mathcal{L}g,g)_{L^2(\mathbb{R}^3_v)}$.

We define a non-isotropic norm associated to the cross-section $b(\cos\theta)$ with respect to the velocity variable as follows

$$|||g|||^2 = \iiint b(\cos\theta)\mu_*\big(g' - g\big)^2 + \iiint b(\cos\theta)g_*^2\big(\sqrt{\mu'} - \sqrt{\mu}\big)^2, \tag{9}$$

where the triple integral is over $\mathbb{R}^3_v \times \mathbb{R}^3_{v_*} \times \mathbb{S}^2_\sigma$. This is a non-isotropic norm as it involves both a derivative of order s and a weight of order s with respect to the velocity variables due to the singularity of cross-section $b(\cos\theta)$, and some kind of mixing between these two effects. A crucial result in our proofs is given by

Lemma 2.1. *There exists $C > 0$ such that, for any $g \in H^s_s(\mathbb{R}^3_v)$*

$$C_1\big(\|g\|^2_{H^s} + \|g\|^2_{L^2_s}\big) \leqslant |||g|||^2 \leqslant C\|g\|^2_{H^s_s}, \tag{10}$$

$$\big|\big(\Gamma(f,g),h\big)_{L^2(\mathbb{R}^3)}\big| \leqslant C\big(\|f\|_{L^2_s}|||g||| + \|g\|_{L^2_s}|||f|||\big)|||h|||. \tag{11}$$

Sketch of the proof. One can show that, for general functions f and g,

$$\iiint bf_*^2\big(g' - g\big)^2 \lesssim \big|\big(Q\big(f^2,g\big),g\big)\big| + \left|\iiint bg^2\big(f_*^{2'} - f_*^2\big)\right|.$$

The upper bound in estimate (10) is then a direct consequence of estimate (8), together from the cancellation Lemma from [1], taking $g = \sqrt{\mu}$. For the lower bound in (10), denote by A and B the two terms that appear in the non-isotropic norm given in (9). By using the proof of Proposition 2 from [1], one can show that

$$A \geqslant c_1\|g\|^2_{H^s} - c_2\|g\|^2_{L^2}.$$

Still using the arguments from [1], $B \sim B_1 + B_2$, where

$$B_1 \sim \iint b\left(\frac{\xi}{|\xi|}.\sigma\right)\widehat{g^2}(0)\big|\widehat{\mu^{1/2}}(\xi_+) - \widehat{\mu^{1/2}}(\xi)\big|^2\,\mathrm{d}\sigma\,\mathrm{d}\xi,$$

$$B_2 \sim \iint b\left(\frac{\xi}{|\xi|}.\sigma\right)\big(\widehat{g^2}(0) - Re\,\widehat{g^2}\big(\xi^-\big)\big)\widehat{\mu^{1/2}}(\xi)\widehat{\mu^{1/2}}(\xi)\,\mathrm{d}\sigma\,\mathrm{d}\xi,$$

with $\xi^\pm = \frac{\xi}{2} \pm \frac{|\xi|}{2}\sigma$ and $\widehat{\cdot}$ designates the Fourier transformation. It follows that

$$B_1 \geqslant c_3\|g\|^2_{L^2}, \qquad B_2 \geqslant c_4\|g|L^2_{L^2_s} - c_5\|g\|^2_{L^2}$$

which by adapting the constants yields the lower bound in (10). Again by the methods of [1] and in particular the use of Fourier transform, one can show that

$$\iiint b(\cos\theta)f_*^2\big(g' - g\big)^2 \leqslant C\|f\|^2_{L^2_s}|||g|||^2$$

for some constant C and test funtions f and g. The proof of the estimate (11) follows.

Next, for $m \in \mathbb{N}$, $\ell \in \mathbb{R}$, we introduce the following space with respect to all variables, where $|||\cdot|||$ is the norm in (9) and recall that $W^{\ell}(v) = \langle v\rangle^{\ell} = (1+|v|^2)^{\ell/2}$

$$\mathcal{B}^m_{\ell}(\mathbb{R}^6_{x,v}) = \left\{ g \in \mathcal{S}'(\mathbb{R}^6_{x,v});\ |||g|||^2_{\mathcal{B}^m_{\ell}(\mathbb{R}^6)} = \sum_{|\alpha|\leqslant m} \int_{\mathbb{R}^3_x} \left|\left|\left| W^{\ell}\partial^{\alpha}_{x,v} g(x,\cdot)\right|\right|\right|^2 \mathrm{d}x < +\infty \right\}.$$

Similarly to Lemma 2.1 and its proof, a number of results follows, as for example the following concerned with the control of weighted derivatives: for any $\ell \geqslant 3$, and $N \geqslant 3$, $N \in \mathbb{N}$, we have, for all $\beta \in \mathbb{N}^6$, $|\beta| \leqslant N$,

$$\left|\left(W^{\ell}\partial^{\beta}_{x,v}\Gamma(f,g), h\right)_{L^2(\mathbb{R}^6_{x,v})}\right| \leqslant C\|f\|_{H^N_{\ell}(\mathbb{R}^6)} |||g|||_{\mathcal{B}^N_{\ell}(\mathbb{R}^6)} |||h|||_{\mathcal{B}^0_0(\mathbb{R}^6)}.$$

Similar bounds are available for commutators of the linear and non-linear operators with the weight W^l. Finally, it is important to note that for any $g \in \mathrm{Ker}(\mathcal{L})^{\perp}$, it follows that $(\mathcal{L}g, g)_{L^2}$ is equivalent to $|||g|||^2$. □

2.2. Energy method

Let $N \geqslant 3$, $N \in \mathbb{N}$, $\ell \geqslant 3$. Then on can show the existence of a local solution: for some $\epsilon_1 > 0$ and $0 < T < +\infty$, if $g_0 \in H^N_{\ell}(\mathbb{R}^6)$ and $\|g_0\|_{H^N_{\ell}(\mathbb{R}^6)} \leqslant \epsilon_1$, then the Cauchy problem (6) admits a solution $g \in L^{\infty}([0,T]; H^N_{\ell}(\mathbb{R}^6)) \cap L^2([0,T]; \mathcal{B}^N_{\ell}(\mathbb{R}^6))$.

Furthermore, if $\mu + \mu^{1/2} g_0 \geqslant 0$, then $g \in L^{\infty}([0,T]; H^N_{\ell}(\mathbb{R}^6))$ is a solution of Cauchy problem (6) satisfying $\mu + \mu^{1/2} g \geqslant 0$ on $[0,T] \times \mathbb{R}^6$. Moreover, by applying the uncertainty principle established in [3,2], $g \in C^{\infty}(]0,T[\times \mathbb{R}^6)$.

To get the global existence, using the arguments from [5], we introduce the macro–micro decomposition $g = Pg + (I-P)g = g_1 + g_2$, $Pg = g_1 = (a + b\cdot v + c|v|^2)\mu^{1/2}$.

The next lemma is similar to the corresponding results in [5].

Lemma 2.2 *(Macro-energy estimate). For $|\alpha| \leqslant N-1$,*

$$\left\|\nabla_x \partial^{\alpha}(a,b,c)\right\|^2_{L^2_x} \leqslant -\frac{\mathrm{d}}{\mathrm{d}t}\left\{\left(\partial^{\alpha} r, \nabla_x\partial^{\alpha}(a,b,c)\right)_{L^2_x} + \left(\partial^{\alpha} b, \nabla_x \partial^{\alpha} a\right)_{L^2_x}\right\} + C\|g_2\|^2_{H^N_x(L^2_v)} + \mathcal{D}_1\mathcal{E}_1. \tag{12}$$

r denotes the previous quantities in term of a, b, c as in [5], $\mathcal{D}_1 = \|\nabla_x(a,b,c)\|^2_{H^{N-1}_x} + \|g_2\|^2_{H^N_x(L^2_v)}$, $\mathcal{E}_1 = \|(a,b,c)\|^2_{H^{N-1}_x} + \|g_2\|^2_{H^{N-1}_x(L^2_v)} = \|g\|^2_{H^{N-1}_x(L^2_v)}$.

For $1 \leqslant |\alpha| \leqslant N$ and $N \geqslant 3$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\|\partial^{\alpha}_x g\right\|^2_{L^2_{\ell}(\mathbb{R}^6)} + \left|\left|\left|\partial^{\alpha}_x g\right|\right|\right|^2_{\mathcal{B}^0_{\ell}(\mathbb{R}^6)} \leqslant \mathcal{E}_3\mathcal{D}_3 + \left\|\partial^{\alpha}_x g_2\right\|^2_{L^2(\mathbb{R}^6)} + \left\|\nabla_x(a,b,c)\right\|^2_{H^{N-1}_x},$$

where $\mathcal{D}_2 = \|\nabla_x(a,b,c)\|^2_{H^{N-1}_x} + |||g_2|||^2_{\mathcal{B}^N_0(\mathbb{R}^6)}$, $\mathcal{E}_2 = \|g\|^2_{H^N_x(L^2_v)} = \|(a,b,c)\|^2_{H^N_x} + \|g_2\|^2_{H^N_x(L^2_v)}$, $\mathcal{E}_3 = \|(a,b,c)\|^2_{H^N_x} + \|g_2\|^2_{H^N_{\ell}(\mathbb{R}^6)}$, $\mathcal{D}_3 = \|\nabla_x(a,b,c)\|^2_{H^{N-1}_x} + |||g_2|||^2_{\mathcal{B}^N_{\ell}(\mathbb{R}^6)}$.

The control of g_2 is then provided by the following lemma, using the complete functional estimates provided in the previous section.

Lemma 2.3. *Letting $|\beta| = |\alpha + \gamma| \leqslant N$, $|\alpha| \leqslant N-1$, $|\gamma| \geqslant 1$, $N \geqslant 3$, one has*

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\left\|\partial^{\beta}_{x,v} g_2\right\|^2_{L^2(\mathbb{R}^6)} + \left|\left|\left|\partial^{\beta}_{x,v} g_2\right|\right|\right|^2_{\mathcal{B}^0_{\ell}(\mathbb{R}^6)} \\ &\quad \leqslant \mathcal{E}_3\mathcal{D}_3 + \|g_2\|^2_{H^{|\beta|}_{\ell}(\mathbb{R}^6)} + |||g_2|||^2_{\mathcal{B}^{|\alpha|+|\gamma|-1}_{\ell}(\mathbb{R}^6)} + \left\|\nabla_x(a,b,c)\right\|^2_{H^{N-1}_x} + \|g_2\|^2_{H^N_x(L^2_v)}.\end{aligned} \tag{13}$$

To conclude the proof of the global existence, we define the instant energy functional

$$\mathcal{E} = \sum_{|\alpha|\leqslant N} \left\|\partial^{\alpha}_x g\right\|^2_{L^2(\mathbb{R}^6)} + \sum_{|\beta|=|\alpha+\gamma|\leqslant N,\ |\alpha|\leqslant N-1, |\gamma|\geqslant 1} \left\|\partial^{\beta}_{x,v} g_2\right\|^2_{L^2_{\ell}(\mathbb{R}^6)}, \tag{14}$$

and setting, for suitable constants $C_{\bullet}^{\bullet}$,

$$\mathcal{D} = \sum_{|\alpha|\leqslant N-1} C^{(1)}_{\alpha}\left\|\nabla_x\partial^{\alpha}_x(a,b,c)\right\|^2_{L^2_x} + \sum_{|\alpha|\leqslant N} C^{(2)}_{\alpha}\left|\left|\left|\partial^{\alpha}_x g_2\right|\right|\right|^2_{\mathcal{B}^0_{\ell}(\mathbb{R}^6)} + \sum_{|\beta|=|\alpha+\gamma|\leqslant N, |\alpha|\leqslant N-1, |\gamma|\geqslant 1} C^{(3)}_{\alpha,\gamma}\left|\left|\left|\partial^{\beta}_{x,v} g_2\right|\right|\right|^2_{\mathcal{B}^0_{\ell}(\mathbb{R}^6)},$$

we get the following a priori estimate, from which one deduces the global existence.

Proposition 2.4. *Let* $N, \ell \geqslant 3$. *Let* $T > 0$ *and suppose that* g *is a classical solution to the Cauchy problem on* $[0, T]$. *There exist constants* $M_0, M_1 > 0$ *such that if* $\max_{0 \leqslant t \leqslant T} \mathcal{E}(t) \leqslant M_0$, *then* g *enjoys the estimate*

$$\mathcal{E}(t) + \int_0^t \mathcal{D}(\tau)\,\mathrm{d}\tau \leqslant M_1 \mathcal{E}(0),$$

for any $t \in [0, T]$.

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