C. R. Acad. Sci. Paris, Ser. I 348 (2010) 915–918

Contents lists available at ScienceDirect

C. R. Acad. Sci. Paris, Ser. I

www.sciencedirect.com

Differential Geometry/Differential Topology

The Witten deformation for even dimensional spaces with cone-like singularities and admissible Morse functions

La déformation de Witten sur des espaces singuliers de dimension paire à singularités coniques

Ursula Ludwig

Mathematisches Institut, Eckerstrasse 1, 79104 Freiburg, Germany

ARTICLE INFO

Article history:
Received 2 March 2010
Accepted after revision 20 July 2010
Available online 5 August 2010

Presented by Jean-Michel Bismut

ABSTRACT

In this Note we generalise the Witten deformation to even dimensional Riemannian manifolds with cone-like singularities X and certain functions f, which we call admissible Morse functions. As a corollary we get Morse inequalities for the L^2-Betti numbers of X. The contribution of a singular point p of X to the Morse inequalities can be expressed in terms of the intersection cohomology of the local Morse datum of f at p. The definition of the class of functions which we study here is inspired by stratified Morse theory as developed by Goresky and MacPherson. However the setting here is different since the spaces considered here are manifolds with cone-like singularities instead of Whitney stratified spaces.

RÉSUMÉ

Le but de cette Note est d'étendre la déformation de Witten au cas d'un espace singulier X de dimension paire à singularités coniques, muni de fonctions appelées fonctions de Morse admissibles. Comme conséquence on obtient des inégalités de Morse pour les nombres de Betti L^2 de X. La contribution d'un point singulier p de X aux inégalités de Morse s'exprime en fonction de la cohomologie d'intersection des données de Morse local. La définition des fonctions de Morse admissibles est inspirée par la théorie de Morse stratifiée de Goresky et MacPherson. Mais ici on travaille sur des espaces singuliers à singularités coniques au lieu d'espaces munis d'une stratification de Whitney.

1. Introduction

Let (X, g) be a Riemannian space of dimension $n =: 2\nu$ with isolated cone-like singularities $\Sigma := \{p_1, \dots, p_N\}$. By this we mean that each $p \in \Sigma$ admits an open neighbourhood $U_\epsilon(p)$ in X such that $(U_\epsilon(p) \setminus \{p\}, g_{|U\setminus\{p\}})$ is isometric to $(\mathrm{cone}_\epsilon(L_p) \setminus \{0\}, \mathrm{d}r^2 + r^2 g_{L_p})$ for some $\epsilon > 0$. Hereby L_p is a smooth compact manifold called the link of X at p, g_{L_p} is a metric on L_p and r denotes the radial coordinate on the cone $\mathrm{cone}_\epsilon(L_p) = [0, \epsilon) \times L_p / \{0\} \times L_p$. We study a certain class of functions on X, which we call admissible Morse functions:

E-mail address: ursula.ludwig@math.uni-freiburg.de.

1631-073X/$ – see front matter
doi:10.1016/j.crma.2010.07.020

Definition 1. Let $f : X \to \mathbb{R}$ be a continuous function which is smooth outside the singularities of X. The function f is called an admissible Morse function if the restriction $f_{|X\setminus\Sigma}$ is Morse in the smooth sense and moreover if for any singular point $p \in \Sigma$ the function f has the following form in local coordinates $(r,\varphi) \in (0,\epsilon) \times L_p$ near p: $f(r,\varphi) = f(p) + rh$, where $h : L_p \to \mathbb{R}$ is a smooth function on the link such that $|\nabla f| = |\nabla h|^2 + h^2 \geqslant a^2$ for some $a > 0$.

The above definition of an admissible Morse function is motivated by stratified Morse theory as developed by Goresky and MacPherson in [7]: A stratified Morse function is not critical in the normal directions for any critical point of a lower dimensional stratum.

Let us choose $0 < \delta \ll \epsilon$. Let $B_\epsilon(p)$ denote the closed ϵ-ball around p. The local Morse datum of f at $p \in \Sigma$ is in our case reduced to the normal local Morse datum and is defined as the pair of spaces

$$\big(B_\epsilon(p) \cap f^{-1}\big(\big[f(p)-\delta, f(p)+\delta\big]\big), B_\epsilon(p) \cap f^{-1}\big(f(p)-\delta\big)\big). \tag{1}$$

Actually it is independent of the choices of ϵ and δ. As in stratified Morse theory (see [7], p. 66) we will call $l_p^- := B_\epsilon(p) \cap f^{-1}(f(p)-\delta)$ the lower halflink of f at p. The local Morse datum is homeomorphic to the pair $(B_\epsilon(p), l_p^-)$. Let us denote by $m_p^i := \dim IH^i(B_\epsilon(p), l_p^-)$, where IH^* denotes intersection cohomology (with middle perversity). Let us denote by $c_i(f_{|X\setminus\Sigma})$ the number of critical points of index i of the restriction $f_{|X\setminus\Sigma}$ and by $c_i(f) := c_i(f_{|X\setminus\Sigma}) + \sum_{p\in\Sigma} m_p^i$.

For Riemannian manifolds with cone-like singularities the intersection cohomology can also be defined analytically, namely as the cohomology of the complex of L^2-forms. This complex computes the so-called L^2-cohomology $H^i_{(2)}(X)$. We denote by $b_i^{(2)}(X) := \dim H^i_{(2)}(X)$ the L^2-Betti numbers of X.

The main goal of this Note is to generalise the Witten deformation (see [11,8]) to the above situation. As a corollary we give an analytic proof of the below Morse inequalities:

Theorem 2. *Let X be a singular Riemannian space as above and let $f : X \to \mathbb{R}$ be an admissible Morse function on X. Then the following Morse inequalities hold*:

$$\sum_{i=0}^{k}(-1)^{k-i}c_i(f) \geqslant \sum_{i=0}^{k}(-1)^{k-i}b_i^{(2)}(X), \quad \textit{for all } 0 \leqslant k < n; \qquad \sum_{i=0}^{n}(-1)^{i}c_i(f) = \sum_{i=0}^{n}(-1)^{i}b_i^{(2)}(X). \tag{2}$$

Theorem 2 is also valid writing $\dim IH^i(X)$ instead of $b_i^{(2)}(X)$ and in this case can be proved geometrically in a standard way by using an appropriate deformation lemma and the corresponding long exact sequence for intersection cohomology. However we are interested here in generalising the Witten deformation to singular spaces. Let us also point out that unlike in [7] here we do not work on general Whitney stratified spaces but on Riemannian manifolds with cone-like singularities. The main reason for this being that for these spaces the complex of L^2-forms and its cohomology are well understood. In the presence of singularities we deform the complex of L^2-forms instead of the de Rham complex. By perturbation techniques one can get the results above for a slightly more general situation, namely for conformally conic Riemannian manifolds (see [4] for a definition) and functions such that the normal form in Definition 1 includes higher order terms in r. The Witten deformation for singular complex algebraic curves (i.e. $n = 2$) has been treated in a previous paper [10]. The proofs of a part of the results presented here (Proposition 3 and Theorem 4) are easy generalisations of the proofs there. The main work consists in understanding the local model operator near the singular points of X; in the situation treated in [10] the local model operator has a simple form and its spectrum and eigenvalues are computed explicitly.

2. The Witten deformation of the complex of L^2-forms and the spectral gap theorem

The de Rham complex of smooth differential forms with compact supports $(\Omega_0^*(X\setminus\Sigma), d, \langle\,,\,\rangle)$ has a unique extension into a Hilbert complex $(\mathcal{C}, d, \langle\,,\,\rangle)$ in the Hilbert space of square integrable forms (see [3] for the definition of a Hilbert complex). Hereby $\langle\,,\,\rangle$ denotes the L^2-metric: $\langle\alpha,\beta\rangle = \int_{X\setminus\Sigma}\alpha\wedge *\beta$. The cohomology of the complex $(\mathcal{C}, d, \langle\,,\,\rangle)$ is the L^2-cohomology of X, $H^i_{(2)}(X)$. The Witten method consists in deforming the complex $(\Omega_0^*(X\setminus\Sigma), d, \langle\,,\,\rangle)$ into a complex $(\Omega_0^*(X\setminus\Sigma), d_t, \langle\,,\,\rangle)$, where $d_t\omega = e^{-tf}\,\mathrm{d}(e^{tf}\omega) = \mathrm{d}\omega + t\,\mathrm{d}f\wedge\omega$. We denote by δ_t the formal adjoint of d_t with respect to the L^2-metric $\langle\,,\,\rangle$.

Proposition 3. *The complex $(\Omega_0^*(X\setminus\Sigma), d_t, \langle\,,\,\rangle)$ has a unique extension into a Hilbert complex $(\mathcal{C}_t, d_t, \langle\,,\,\rangle)$. Moreover the associated Laplacian $\Delta_t = d_t\delta_t + \delta_t d_t$ with* $\mathrm{dom}(\Delta_t) = \{\Psi \in \mathrm{L}^2(\Lambda^*(T^*(X\setminus\Sigma))) \mid d_t\Psi, \delta_t\Psi, d_t\delta_t\Psi, \delta_t d_t\Psi \in \mathrm{L}^2(\Lambda^*(T^*(X\setminus\Sigma)))\}$ *is a non-negative, self-adjoint, discrete operator. Moreover* $\ker(\Delta_t^{(i)}) \simeq H^i((\mathcal{C}_t, d_t, \langle\,,\,\rangle)) \simeq H^i_{(2)}(X)$, *where $\Delta_t^{(i)}$ denotes the restriction of Δ_t to i-forms.*

The proof is a generalisation of [10]. We call the operator Δ_t, with $\mathrm{dom}(\Delta_t)$, the Witten Laplacian.

Theorem 4 *(Spectral gap theorem).*

(a) *There exist constants* C_1, C_2, C_3 *and* $t_0 > 0$ *depending on* X *and* f *such that* $\operatorname{spec}(\Delta_t) \cap (C_1 e^{-C_2 t}, C_3 t) = \emptyset$ *for any* $t > t_0$.
(b) *Let us denote by* $(\mathbb{F}_t, d_t, \langle\, ,\, \rangle)$ *the subcomplex of* $(\mathcal{C}_t, d_t, \langle\, ,\, \rangle)$ *generated by all eigenforms of the Witten Laplacian* Δ_t *to eigenvalues in* $[0, 1]$. *Then there exists* $t_0 > 0$ *such that, for* $t \geqslant t_0$,

$$\dim \mathbb{F}_t^i = c_i(f_{|X\setminus\Sigma}) + \sum_{p\in\Sigma} m_p^i =: c_i(f). \tag{3}$$

Once the local situation near singular points of X is understood, one can proceed as in the smooth case to prove Theorem 4 (see e.g. [1], Section 9). The Morse inequalities in Theorem 2 follow from the spectral gap theorem (part (b)) by the usual argument.

The basic step in the proof of the spectral gap theorem is the study of a model operator for the Witten Laplacian in the neighbourhood of a singular point $p \in \Sigma$ of X. Let us denote by $(\operatorname{cone}(L_p), \mathrm{d}r^2 + r^2 g_{L_p})$ the infinite cone over the link L_p. Let $f = rh$ be a function defined on the whole infinite cone, with $h : L_p \to \mathbb{R}$, $|\nabla h|^2 + h^2 \geqslant a^2 > 0$. Let $(\Omega_0^*(\operatorname{cone}(L_p)), d, \langle\, ,\, \rangle)$ be the de Rham complex of smooth compactly supported differential forms on the infinite cone. We denote by $(\Omega_0^*(\operatorname{cone}(L_p)), d_t, \langle\, ,\, \rangle)$ the complex obtained by deforming the complex $(\Omega_0^*(\operatorname{cone}(L_p)), d, \langle\, ,\, \rangle)$ by means of the function f, i.e. $d_t\omega := e^{-tf}\,\mathrm{d}(e^{tf}\omega)$. As before one can show that there is a unique Hilbert complex $(\mathcal{D}_t, d_t, \langle\, ,\, \rangle)$ extending the complex $(\Omega_0^*(\operatorname{cone}(L_p)), d_t, \langle\, ,\, \rangle)$. We define the model Witten Laplacian $\Delta_{t,p}$ as the Laplacian associated to the Hilbert complex $(\mathcal{D}_t, d_t, \langle\, ,\, \rangle)$.

Theorem 5 *(Local spectral gap theorem).*

(a) *There exists* $c > 0$ *such that, for* t *large enough,* $\operatorname{spec}(\Delta_{t,p}) \subset \{0\} \cup [ct^2, \infty)$. *Moreover all forms in* $\ker(\Delta_{t,p})$, *as well as their derivatives have exponential decay outside a small neighbourhood of the singularity.*
(b) *One has, for* t *large enough,*

$$\ker\big(\Delta_{t,p}^{(i)}\big) \simeq IH^i\big(B_\epsilon(p), l_p^-\big). \tag{4}$$

The corresponding local spectral gap theorem for curves in [10] has been proved by an explicit computation. There the local model operator is simply "$\Delta_{t,p} = \Delta_p + t^2$" and the eigenforms in $\ker(\Delta_{t,p})$ are given explicitly in terms of modified Bessel functions, which have exponential decay for $r \to \infty$. In the general situation treated here the model operator has the form $\Delta_{t,p} = \Delta_p + tM_f + t^2|\nabla f|^2$, where M_f is a 0-order operator having a pole of order 1 in r at $r \to 0$. However one can show the following proposition, using the line of arguments in [4] (where the analogous statement for the Laplacian is shown):

Proposition 6. *Let us denote by* $\Delta_{t,p}^{\mathcal{F}}$ *the Friedrichs extension of* $\Delta_{t,p|\Omega_0^*}$. *Then, for* $k \neq \nu$: $\Delta_{t,p}^{(k)} = \Delta_{t,p}^{(k),\mathcal{F}}$.

Using further techniques from [5], Section 3, one can give estimates for the Friedrichs extension $\Delta_{t,p}^{\mathcal{F}}$ which will show the local spectral gap theorem for all degrees $k \neq \nu$. The case $k = \nu$ follows using the Hodge decomposition for the local complex $(\mathcal{D}_t, d_t, \langle\, ,\, \rangle)$. To prove the exponential decay of forms in $\ker(\Delta_{t,p})$ one uses Agmon type estimates (similarly to those in e.g. [9], p. 22 ff.).

To prove the isomorphism in part (b) we adapt a "cone construction" in [2] (see also [6]).

Note that in contrast to [10] the proof of Theorem 5 does not give the eigenforms in $\ker(\Delta_{t,p})$ explicitly. Moreover the contribution of the singularity is not concentrated in one degree only (as it is in the curve case) and it depends highly on the chosen admissible function (as can be seen from part (b)). To be able to produce a combinatorial complex in the higher dimensional case it would be therefore necessary to focus on special cases of admissible Morse functions.

Acknowledgements

I wish to thank Jean Michel Bismut for suggesting work on the subject. I am grateful to Jochen Brüning and Jean-Paul Brasselet for many helpful conversations, as well as to Olaf Post. Part of this work was done while I was supported by SFB 647.

References

[1] J.-M. Bismut, G. Lebeau, Complex immersions and Quillen metrics, Publ. Math. Inst. Hautes Etudes Sci. 74 (1991) 1–297.
[2] M. Braverman, V. Silantyev, Kirwan–Novikov inequalities on a manifold with boundary, Trans. Amer. Math. Soc. 358 (8) (2006) 3329–3361.
[3] J. Brüning, M. Lesch, Hilbert complexes, J. Funct. Anal. 108 (1) (1992) 88–132.
[4] J. Brüning, M. Lesch, Kähler–Hodge theory for conformal complex cones, Geom. Funct. Anal. 3 (5) (1993) 439–473.
[5] J. Brüning, M. Lesch, On the spectral geometry of algebraic curves, J. Reine Angew. Math. 474 (1996) 25–66.

[6] M. Farber, E. Shustin, Witten deformation and polynomial differential forms, Geom. Dedicata 80 (1–3) (2000) 125–155.
[7] M. Goresky, R. MacPherson, Stratified Morse Theory, Ergebnisse der Mathematik und ihrer Grenzgebiete (3) (Results in Mathematics and Related Areas (3)), vol. 14, Springer-Verlag, Berlin, 1988.
[8] B. Helffer, J. Sjöstrand, Puits multiples en mécanique semi-classique. IV : Étude du complexe de Witten, Comm. Partial Differential Equations 10 (3) (1985) 245–340.
[9] B. Helffer, Semi-Classical Analysis for the Schrödinger Operator and Applications, Lecture Notes in Mathematics, vol. 1336, Springer-Verlag, 1988.
[10] U. Ludwig, The Witten complex for singular spaces of dimension two with cone-like singularities, Math. Nachr., in press.
[11] E. Witten, Supersymmetry and Morse theory, J. Differential Geom. 17 (4) (1982) 661–692.