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Topology

The homomorphisms between the Dickson–Mùi algebras as modules over the Steenrod algebra ☆

Homorphismes entre l'algèbre de Dickson–Mùi comme module sur l'algèbre de Steenrod

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ABSTRACT

The Dickson–Mùi algebra consists of all invariants in the mod p cohomology of an elementary abelian p-group under the general linear group. It is a module over the Steenrod algebra, $\mathcal{A}$. We determine explicitly all the $\mathcal{A}$-module homomorphisms between the Dickson–Mùi algebras and all the $\mathcal{A}$-module automorphisms of these algebras.

RÉSUMÉ

L'algèbre de Dickson–Mùi consiste en les invariants sous l'action du groupe linéaire dans l'algèbre de cohomologie modulo p d'un p-groupe abélien élémentaire. C'est un module sur l'algèbre de Steenrod $\mathcal{A}$. Nous déterminons explicitement tous les homorphismes $\mathcal{A}$-linéaires entre ces algèbres ainsi que leurs automorphismes ($\mathcal{A}$-linéaires).

1. Statement of results

Let $\mathbb{V} = \mathbb{V}_s$ be an elementary abelian p-group of rank s, where p is a prime. Then $\mathbb{V}$ can also be regarded as an s-dimensional vector space over $\mathbb{F}_p$, the prime field of p elements. Let $H^*(\mathbb{V})$ denote the mod p cohomology of (a classifying space $B\mathbb{V}$ of) the group $\mathbb{V}$. As it is well known

$$H^*(\mathbb{V}) \cong \begin{cases} \mathbb{F}_2[x_1, \ldots, x_s], & p = 2, \\ E(e_1, \ldots, e_s) \otimes \mathbb{F}_p[x_1, \ldots, x_s], & p > 2. \end{cases}$$

Here $(x_1, \ldots, x_s)$ is a basis of $H^1(\mathbb{V}) = Hom(\mathbb{V}, \mathbb{F}_p)$ when $p = 2$, or a basis of $H^2(\mathbb{V})$ and $x_i = \beta(e_i)$ for $1 \leqslant i \leqslant s$ with β the Bockstein homomorphism when $p > 2$.

The general linear group $GL(\mathbb{V}) \cong GL(s, \mathbb{F}_p)$ acts regularly on $\mathbb{V}$ and therefore on $H^*(\mathbb{V})$. The Dickson algebra, which was first studied and explicitly computed by L.E. Dickson [3], is the algebra of all invariants of $\mathbb{F}_p[x_1, \ldots, x_s]$ under the action of $GL(\mathbb{V})$. The invariant algebra $H^*(\mathbb{V})^{GL(\mathbb{V})}$ was explicitly computed by H. Mùi [9] for $p > 2$. We call $H^*(\mathbb{V})^{GL(\mathbb{V})}$ the Dickson–Mùi algebra and denote it by $D(\mathbb{V})$, or simply by D_s, in the both cases $p = 2$ and p an odd prime.

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Being the cohomology of the classifying space $B\mathbb{V}$, the group $H^*(\mathbb{V})$ is equipped with a structure of module over the mod p Steenrod algebra, $\mathcal{A} = \mathcal{A}_p$. The actions of $GL(\mathbb{V})$ and $\mathcal{A}$ upon $H^*(\mathbb{V})$ commute with each other. Therefore, the Dickson–Mùi algebra inherits a structure of module over the Steenrod algebra from $H^*(\mathbb{V})$.

Let $\overline{D}(\mathbb{V})$ or $\overline{D}_s$ be the augmentation ideal of all positive degree elements in the Dickson–Mùi algebra $D_s = D(\mathbb{V})$. We call it the reduced Dickson–Mùi algebra. Let $tr_{n,r} : \overline{D}_n \to \overline{D}_r$ and $Res_{s,n} : \overline{D}_s \to \overline{D}_n$ denote the transfer and the restriction on the Dickson–Mùi algebra that will be defined in detail in Section 2 respectively. Let $\mathbb{U}$ and $\mathbb{W}$ be respectively $\mathbb{F}_p$-vector spaces of dimensions r and n. The above two homomorphisms are also denoted by $tr_{\mathbb{W},\mathbb{U}} : \overline{D}(\mathbb{W}) \to \overline{D}(\mathbb{U})$ and $Res_{\mathbb{V},\mathbb{W}} : \overline{D}(\mathbb{V}) \to \overline{D}(\mathbb{W})$ respectively.

Theorem 1.1. *The $\mathcal{A}$-module homomorphisms*

$$\left\{tr_{n,r}Res_{s,n} \mid 1 \leqslant n \leqslant \min\{r, s\}\right\}$$

form a basis of the vector space $Hom_{\mathcal{A}}(\overline{D}_s, \overline{D}_r)$ of all $\mathcal{A}$-module homomorphisms from $\overline{D}_s$ to $\overline{D}_r$. In particular, $\dim_{\mathbb{F}_p} Hom_{\mathcal{A}}(\overline{D}_s, \overline{D}_r) = \min\{r, s\}$.

The main ingredients of our proof are as follows: Let $\mathbb{U}$ and $\mathbb{V}$ be $\mathbb{F}_p$-vector spaces of dimensions r and s respectively.

First, according to a theorem by Carlsson [2] for $p = 2$ and by Miller [8] for p odd prime, $H^*(\mathbb{U})$ is injective in the category of unstable $\mathcal{A}$-modules. Hence, each $\mathcal{A}$-module homomorphism $f : \overline{D}(\mathbb{V}) \to \overline{D}(\mathbb{U}) \subset H^*(\mathbb{U})$ can be extended to an $\mathcal{A}$-module homomorphism $\hat{f} : H^*(\mathbb{V}) \to H^*(\mathbb{U})$. Secondly, by a theorem of Adams, Gunawardena and Miller [1], $\hat{f}$ can be expressed as

$$\hat{f} = \lambda_1\varphi_1^* + \cdots + \lambda_k\varphi_k^*,$$

where $\lambda_i \in \mathbb{F}_p$ and φ_i^* is the homomorphism induced in cohomology by some linear map $\varphi_i : \mathbb{U} \to \mathbb{V}$ for any i. Then φ_i^* is a homomorphism of $\mathcal{A}$-algebras. Finally, the restrictions and the transfers are taken into account when we recognize the relation between the terms $\lambda_i\varphi_i^*$'s, especially in case $Im(\varphi_i^*) \neq H^*(\mathbb{U})$.

By means of the $\mathcal{A}$-module decomposition $D_s = \mathbb{F}_p \cdot 1 \oplus \overline{D}_s$, we get the following:

Corollary 1.2. *The $\mathcal{A}$-module homomorphisms*

$$\left\{tr_{n,r}Res_{s,n} \mid 0 \leqslant n \leqslant \min\{r, s\}\right\}$$

form a basis of the vector space $Hom_{\mathcal{A}}(D_s, D_r)$ of all $\mathcal{A}$-module homomorphisms from D_s to D_r. In particular, $\dim_{\mathbb{F}_p} Hom_{\mathcal{A}}(D_s, D_r) = \min\{r, s\} + 1$.

Note that $tr_{0,r}Res_{s,0}$ simply maps $1 \in D_s$ to $1 \in D_r$ and vanishes on $\overline{D}_s$.

Let us study the map

$$\theta : Hom(\mathbb{U}, \mathbb{V}) \to Hom_{\mathcal{A}}\big(D(\mathbb{V}), D(\mathbb{U})\big),$$
$$\varphi \mapsto \sum_{g \in GL(\mathbb{U}^*)/GL(\mathbb{U}^*, \varphi^* H^1(\mathbb{V}))} g\varphi^*,$$

where g runs over a set of left coset representatives of $GL(\mathbb{U}^*, \varphi^* H^1(\mathbb{V}))$ in $GL(\mathbb{U}^*)$. Here $GL(\mathbb{U}^*, \varphi^* H^1(\mathbb{V}))$ denotes the subgroup of $GL(\mathbb{U}^*)$ consisting of all isomorphisms $\mathbb{U}^* \to \mathbb{U}^*$ that map $\varphi^* H^1(\mathbb{V})$ to itself. By using Definition 2.1 of transfer, we get $\theta(\varphi) = tr_{\mathbb{W},\mathbb{U}}Res_{\mathbb{V},\mathbb{W}}$, where $\mathbb{W}$ denotes the dual $\mathbb{F}_p$-vector space of $\varphi^* H^1(\mathbb{V})$. Obviously, $Ker(\varphi) = \{u \mid \langle u, \varphi^* H^1(\mathbb{V})\rangle = 0\}$. Hence, we observe that $\theta(\varphi) = \theta(\psi)$ for $\varphi, \psi \in Hom(\mathbb{U}, \mathbb{V})$ if and only if $Ker\varphi \cong Ker\psi$, or equivalently $Im\varphi \cong Im\psi$. We write $\varphi \sim \psi$ to say that this condition is valid. It is easy to see that $(Hom(\mathbb{U}, \mathbb{V})/\sim) \cong GL(\mathbb{V})\backslash Hom(\mathbb{U}, \mathbb{V})/GL(\mathbb{U})$.

Theorem 1.1 and Corollary 1.2 can be re-expressed in the following formulation: The map θ induces two isomorphisms of vector spaces

$$\mathbb{F}_p\big[GL(\mathbb{V})\backslash Hom(\mathbb{U}, \mathbb{V})/GL(\mathbb{U})\big] \xrightarrow{\cong} Hom_{\mathcal{A}}\big(D(\mathbb{V}), D(\mathbb{U})\big),$$
$$\mathbb{F}_p\big[GL(\mathbb{V})\backslash Hom(\mathbb{U}, \mathbb{V})/GL(\mathbb{U})\big]/\mathbb{F}_p 0 \xrightarrow{\cong} Hom_{\mathcal{A}}\big(\overline{D}(\mathbb{V}), \overline{D}(\mathbb{U})\big).$$

In order to get the second isomorphism from the first one, we observe that $\theta(\mathbb{F}_p 0)$ is exactly the subspace of homomorphisms that vanish on $\overline{D}(\mathbb{V})$.

For $\mathbb{U} = \mathbb{V}$, the map θ induces two isomorphisms of algebras

$$\mathbb{F}_p\big[GL(\mathbb{V})\backslash End(\mathbb{V})/GL(\mathbb{V})\big] \xrightarrow{\cong} End_{\mathcal{A}}\big(D(\mathbb{V})\big),$$
$$\mathbb{F}_p\big[GL(\mathbb{V})\backslash End(\mathbb{V})/GL(\mathbb{V})\big]/\mathbb{F}_p 0 \xrightarrow{\cong} End_{\mathcal{A}}\big(\overline{D}(\mathbb{V})\big).$$

Note added in proof. The following problem is probably something of interest.

Problem. Find the conditions on subgroups G of $GL(\mathbb{U})$ and H of $GL(\mathbb{V})$ respectively, under which there is an isomorphism of $\mathbb{F}_p$-vector spaces

$$\mathbb{F}_p\big[H\backslash Hom(\mathbb{U},\mathbb{V})/G\big] \cong Hom_{\mathcal{A}}\big(H^*\big(\mathbb{V}\big)^H, H^*(\mathbb{U})^G\big),$$

and an isomorphism of algebras

$$\mathbb{F}_p\big[H\backslash End(\mathbb{V})/H\big] \cong End_{\mathcal{A}}\big(H^*(\mathbb{V})^H\big).$$

Note that these isomorphisms happen for $G = \{1\}$, $H = \{1\}$ by the theorem of Adams–Gunawardena–Miller, and for $G = GL(\mathbb{U})$, $H = GL(\mathbb{V})$ by the main result of this note.

The commutativity relation of the transfer and the restriction is given as follows:

Proposition 1.3.

(i) $Res_{n,r}tr_{s,n} = tr_{s-n+r,r}Res_{s,s-n+r}$, *for* $n \geqslant \max\{r,s\}$.
(ii) $tr_{n,r}Res_{s,n} = Res_{s-n+r,r}tr_{s,s-n+r}$, *for* $n \leqslant \min\{r,s\}$.

Theorem 1.4. *Let $\mathbb{F}_p[t]$ be the polynomial algebra on an indeterminate t. There are isomorphisms of algebras*

(i) $End_{\mathcal{A}}(D_s) \cong \mathbb{F}_p[t]/(t^{s+1})$,
(ii) $End_{\mathcal{A}}(\overline{D}_s) \cong \mathbb{F}_p[t]/(t^s)$,

which send $tr_{s-1,s}Res_{s,s-1}$ to t.

The vector space $Hom_{\mathcal{A}}(D_s, D_r)$ is equipped with a bimodule structure: It is a right module over $End_{\mathcal{A}}(D_s)$ and a left module over $End_{\mathcal{A}}(D_r)$. By passing to the quotient, $Hom_{\mathcal{A}}(\overline{D}_s, \overline{D}_r)$ is also a bimodule: a right module over $End_{\mathcal{A}}(\overline{D}_s)$ and a left module over $End_{\mathcal{A}}(\overline{D}_r)$. Set $u_i = tr_{\min(r,s)-i,r}Res_{s,\min(r,s)-i}$ for $i \geqslant 0$. Denote $t = tr_{s-1,s}Res_{s,s-1}$ in $End_{\mathcal{A}}(D_s)$ or in $End_{\mathcal{A}}(\overline{D}_s)$, and $t' = tr_{r-1,r}Res_{r,r-1}$ in $End_{\mathcal{A}}(D_r)$ or in $End_{\mathcal{A}}(\overline{D}_r)$.

Proposition 1.5. *The structures of the bimodules $Hom_{\mathcal{A}}(D_s, D_r) \cong \bigoplus_{i=0}^{\min(r,s)} \mathbb{F}_p u_i$ and $Hom_{\mathcal{A}}(\overline{D}_s, \overline{D}_r) \cong \bigoplus_{i=0}^{\min(r,s)-1} \mathbb{F}_p u_i$ are given by*

(i) $u_i t = u_{i+1}$,
(ii) $t' u_i = u_{i+1}$,

where $u_{\min(r,s)+1} = 0$ in $Hom_{\mathcal{A}}(D_s, D_r)$ and $u_{\min(r,s)} = 0$ in $Hom_{\mathcal{A}}(\overline{D}_s, \overline{D}_r)$.

Theorem 1.6. *An $\mathcal{A}$-module endomorphism $f : \overline{D}_s \to \overline{D}_s$ is an automorphism if and only if*

$$f = \lambda id_{D_s} + \sum_{n=1}^{s-1} \lambda_n tr_{n,s}Res_{s,n} \quad (\lambda_n \in \mathbb{F}_p),$$

where λ is a non-zero scalar. In particular, there are exactly $(p-1)p^{s-1}$ automorphisms of the $\mathcal{A}$-module $\overline{D}_s$.

Theorem 1.7. *If an $\mathcal{A}$-module endomorphism $f : \overline{D}_s \to \overline{D}_s$ is non-zero on the least positive degree generator of the Dickson–Mùi algebra, then it is an automorphism.*

Corollary 1.8. *The reduced Dickson–Mùi algebra is an indecomposable module over the Steenrod algebra.*

From this result, the problem of classifying the indecomposable modules over the Steenrod algebra should be of interest.

Theorem 1.9. *Let $f : \overline{D}_s \to \overline{D}_r$ be a homomorphism of $\mathcal{A}$-algebras. Then*

$$f = \begin{cases} \lambda Res_{s,r}, & r \leqslant s, \\ 0, & r > s, \end{cases}$$

where λ is a scalar.

Corollary 1.10. *Let* $End^{alg}_{\mathcal{A}}(\overline{D}_s)$ *be the algebra of all* $\mathcal{A}$*-algebra endomorphisms of* $\overline{D}_s$*. Then there is an isomorphism of algebras* $End^{alg}_{\mathcal{A}}(\overline{D}_s) \cong \mathbb{F}_p$.

2. Transfer and restriction on the Dickson–Mùi algebras

Let $\mathbb{U}$ and $\mathbb{V}$ be $\mathbb{F}_p$-vector spaces of respectively dimensions r and s with $r \leqslant s$. Then V can be regarded as a direct sum $\mathbb{V} = \mathbb{U} \oplus \mathbb{V}'$ of $\mathbb{U}$ and some vector space $\mathbb{V}'$. Therefore, $H^*(\mathbb{U})$ is thought of as a subalgebra of $H^*(\mathbb{V})$. Recall that

$$H^*(\mathbb{U}) \cong \begin{cases} S(\mathbb{U}^*), & p = 2, \\ E(\mathbb{U}^*) \otimes S(\beta\mathbb{U}^*), & p > 2, \end{cases}$$

where $S(\mathbb{X})$ and $E(\mathbb{X})$ denote the symmetric algebra and the exterior algebra on the vector space $\mathbb{X}$, with β the Bockstein homomorphism for $p > 2$.

If $g : \mathbb{V}^* \to \mathbb{V}^*$ is a linear isomorphism, then $g = g|_{\mathbb{U}^*}$ is an isomorphism from $\mathbb{U}^*$ to $g\mathbb{U}^*$. It also gives rise to an isomorphism $g : H^*(\mathbb{U}) \to gH^*(\mathbb{U})$. Let $GL(\mathbb{V}^*, \mathbb{U}^*)$ denote the subgroup of $GL(\mathbb{V}^*)$ consisting of all isomorphisms $\mathbb{V}^* \to \mathbb{V}^*$ that map $\mathbb{U}^*$ to itself.

Definition 2.1. The transfer $tr_{\mathbb{U},\mathbb{V}} : H^*(\mathbb{U})^{GL(\mathbb{U})} \to H^*(\mathbb{V})^{GL(\mathbb{V})}$ is given by

$$tr_{\mathbb{U},\mathbb{V}}(Q) = \sum_{g \in GL(\mathbb{V}^*)/GL(\mathbb{V}^*,\mathbb{U}^*)} gQ,$$

for $Q \in H^*(\mathbb{U})^{GL(\mathbb{U})}$, where g runs over a set of left coset representatives of $GL(\mathbb{V}^*, \mathbb{U}^*)$ in $GL(\mathbb{V}^*)$.

Definition 2.2. The *restriction* from $D(\mathbb{V})$ to $D(\mathbb{U})$, denoted by $Res_{\mathbb{V},\mathbb{U}} : D(\mathbb{V}) \to D(\mathbb{U})$ or by $Res_{s,r} : D_s \to D_r$ for $r = \dim \mathbb{U} \leqslant \dim \mathbb{V} = s$, is the homomorphism $i^*_{\mathbb{U},\mathbb{V}} : H^*(\mathbb{V})^{GL(\mathbb{V})} \to H^*(\mathbb{U})^{GL(\mathbb{U})}$ induced by an inclusion $i_{\mathbb{U},\mathbb{V}} : \mathbb{U} \to \mathbb{V}$.

The restriction $Res_{\mathbb{V},\mathbb{U}}$ does not depend on the choice of the inclusion $i_{\mathbb{U},\mathbb{V}}$.

The contains of this note will be published in detail elsewhere.

3. Final remarks

There is some overlap between Kechagias' manuscript [7] and this note, such as the dimension of $End_{\mathcal{A}}(\overline{D}_s)$ and the indecomposability of $\overline{D}_s$.

However, it should be noted that the results of the note are general, more precise, while our proof is less technical and more conceptual.

Our proof is essentially based on the results of the papers [4–6].

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