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Mathematical Analysis

How likely is Buffon's ring toss to intersect a planar Cantor set?

Quelles sont les chances pour un cercle de Buffon lancé sur le plan de faire l'intersection avec une voisinage d'un ensemble de Cantor?

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ABSTRACT

In Bateman and Volberg (2008) [1], it was shown that the n-th partial 1/4 Cantor in the plane set decays in Favard length no faster than $C\frac{\log n}{n}$. In Bond and Volberg (2008) [2], the so-called circular Favard length of the same set is studied, and the same estimate is shown to persist when the circle has radius $r \geqslant Cn$. By considering characteristic functions, the result of Bond and Volberg (2008) [2] naturally leads to a conjecture which (if true) would imply the sharpness of the $L \log\log L$ boundedness of the circular maximal operator proved by Seeger, Tao and Wright (2005) [3].

RÉSUMÉ

Dans Bateman et Volberg (2008) [1], on a démontré que la longueur de Favard de la stage n-ième d'ensemble 1/4 de Cantor décroit au plus comme $C\frac{\log n}{n}$. Dans Bond et Volberg (2008) [2], on a introduit une longueur circulaire de Favard, et on a démontré que les même estimations sont valable, au moins si le rayon du cercle satisfait $r \geqslant Cn$. Le résulat de Bond et Volberg (2008) [2] mene naturallement à une hypothèse qui (si soit valable) donne la preuve que le résultat concernant la fonction maximale circulaire de Seeger, Tao et Wright (2005) [3] est exact.

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1. Definitions

The four-corner Cantor set $\mathcal{K}$ is constructed by replacing the unit square by four sub-squares of side length 1/4 at its corners, and iterating this operation in a self-similar manner in each sub-square. After the nth iteration of the similarity maps, let us call the resulting set $\mathcal{K}_n$.

The *Favard length*, or *Buffon needle probability*, of a planar set E is defined by

$$Fav(E) = \frac{1}{\pi}\int_0^\pi \left|Proj_\theta(E)\right| \mathrm{d}\theta, \tag{1}$$

where $Proj_\theta$ denotes the orthogonal projection from $\mathbb{R}^2$ to direction with angle θ, and $|A|$ denotes the Lebesgue measure of a measurable set $A \subset \mathbb{R}$.

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In [2], a related *circular Favard length*, or *Buffon noodle probability*, was studied. To get circular Favard length Fav_σ instead of usual Favard length *Fav*, orthogonal projection along the line is replaced by projection along a circular arc tangent to the line. Specifically, define

$$F_r(y) := r - \sqrt{r^2 - y^2}. \tag{2}$$

Also define $\sigma_0(x, y) := (x - F_r(y), y)$, and $\sigma_\theta := R_{-\theta} \circ \sigma_0 \circ R_\theta$, where R_θ is clockwise rotation by the angle θ.[1] Finally, let

$$Fav_\sigma(\mathcal{K}_n) := \frac{1}{\pi}\int_0^\pi \left|Proj_\theta\big(\sigma_\theta(\mathcal{K}_n)\big)\right| \mathrm{d}\theta.$$

For any Cantor square $Q \subset \mathcal{K}_n$, let $\chi_{Q,\theta} := \chi_{Proj_\theta(\sigma_\theta(Q))}$.

2. The result and the main approach

One way of studying Favard length of structured discrete sets like $\mathcal{K}_n$ is through a certain projection multiplicity function $f_{n,\theta,\sigma}$ (as used in [1,2], and others):

$$f_{n,\theta,\sigma} := \sum_{\text{Cantor squares } Q \subset \mathcal{K}_n} \chi_{Q,\theta}.$$

This is because $Proj_\theta(\sigma_\theta(\mathcal{K}_n)) = supp(f_{n,\theta,\sigma})$, which we will also call $E_{n,\theta,\sigma}$. The idea is that as the similarities are iterated, the squares stack in a self-similar manner, and the L^2 norms of $f_{n,\theta,\sigma}$ should grow, while $|E_{n,\theta,\sigma}|$ should decrease. However, the Cauchy inequality describes a limitation on this effect: for any fixed interval of angles I,

$$\int_I |E_{n,\theta,\sigma}| \geqslant \frac{(\int_I \int_{\mathbb{R}} f_{n,\theta,\sigma}\, \mathrm{d}x\, \mathrm{d}\theta)^2}{(\int_I \int_{\mathbb{R}} f^2_{n,\theta,\sigma}\, \mathrm{d}x\, \mathrm{d}\theta)}. \tag{3}$$

The idea is to pick $\approx \log n$ many disjoint intervals I_j such that each such estimate gives

$$\int_{I_j} |E_{n,\theta,\sigma}|\, \mathrm{d}\theta \geqslant \frac{C}{n}. \tag{4}$$

Summing over j, the result will be:

Theorem 2.1. *For each $c > 0$, there exists $C > 0$ such that whenever $r \geqslant cn$, $Fav_\sigma(\mathcal{K}_n) \geqslant C\frac{\log n}{n}$. Further, we may interpret $Fav(\mathcal{K}_n)$ to be $Fav_\sigma(\mathcal{K}_n)$ in the case $r = \infty$.*

Good intervals I_j can be found near $\theta = \arctan(1/2)$, because on this direction, $\mathcal{K}_n$ orthogonally projects onto a single connected interval, and the projected squares intersect only on their endpoints. These almost-disjoint projected intervals induce a 4-adic structure on the interval. Let us rotate the axes and redefine the old $\arctan(1/2)$ direction to be our new $\theta = 0$ direction.

We will then let $I_j := [\arctan(4^{-j-1}), \arctan(4^{-j})]$, $3 < j < \log n$. Then $I_{\log n}$ will be the closest direction to 0, and it's reasonable to think that on average, each time j decreases by 1, I_j will grow by the factor 4, and $|E_{n,\theta,\sigma}|$ will decay no more than by a factor of $1/4$, resulting in estimate (4).

Trivially, $[\int_{I_j}\int f_{n,\theta,\sigma}\, \mathrm{d}x\, \mathrm{d}\theta]^2 \leqslant C4^{-2j}$, while

$$f^2_{n,\theta,\sigma} = \sum_{Q,Q'} \chi_{Q,\theta}\chi_{Q',\theta} = \sum_{Q \neq Q'} \chi_{Q,\theta}\chi_{Q',\theta} + \sum_Q \chi^2_{Q,\theta}.$$

Integrating over $I_j \times \mathbb{R}$, the latter diagonal sum becomes $C4^{-j} \leqslant Cn4^{-2j}$ (the inequality uses $j < \log n$). When estimating the other integral, things become combinatorial – most of these terms are identically 0 in $I_j \times \mathbb{R}$. So define $A_{j,k}$ to be the set of pairs $P = (Q, Q')$ of Cantor squares such that there exists $\theta \in [0, \pi]$ such that the σ_θ images of the centers q and q' of Q and Q' have vertical distance $4^{-k-1} \leqslant |y_{\sigma_\theta(q)} - y_{\sigma_\theta(q')}| \leqslant 4^{-k}$ and satisfy the condition on horizontal spacing

$$4^{-j-1} \leqslant \left|\frac{x_{\sigma_\theta(q)} - x_{\sigma_\theta(q')}}{y_{\sigma_\theta(q)} - y_{\sigma_\theta(q')}}\right| \leqslant 4^{-j}. \tag{5}$$

[1] Note that if we replace σ with the identity map, we are in the setting of [1]. We will often appeal to the $\sigma = Id$ case for intuition, while noting that the content of [2] is that the arguments of [1] carry over into [2] when $cn \leqslant r < \infty$ with the only difference being a change in the universal constants.

We can think of 4^{-j} as being $\tan(\theta)$ for θ such that the σ_θ images of the squares Q, Q' have overlap in the projection onto θ. In [1], it was proved that

$$|A_{j,k}| \leqslant C4^{2n-k-2j}, \tag{6}$$

when $r = \infty$. To get the same estimate for $cn \leqslant r < \infty$ as shown in [2], it suffices to compare the two cases with an application of the following lemma[2]:

Lemma 2.2. *Let $\varepsilon > 0$ be small enough. Let $T : \mathbb{C} \to \mathbb{C}$ be such that $Lip(T - Id) < \varepsilon$. Then $\forall z, w \in \mathbb{C}$,*

$$\left|\arg(z-w) - \arg\bigl(T(z) - T(w)\bigr)\right| < 2\varepsilon (mod\, 2\pi).$$

This is where the condition $r > cn$ is used: to make the lemma sufficient for the purposes of relation 5. For any $P = (Q, Q') \in A_{j,k}$, it suffices to have the integral $\nu_P := \int_0^\pi \int_{\mathbb{R}} \chi_{Q,\theta}\chi_{Q',\theta}\,dx\,d\theta$ satisfy the estimate

$$\nu_P \leqslant C4^{k-2n}, \tag{7}$$

since the integrand is supported only for angles belonging to I_{j-1}, I_j, and I_{j+1}. So we fix j and sum over k to get

$$\int\limits_{I_j \times \mathbb{R}} \sum_{Q \neq Q'} \chi_{Q,\theta}\chi_{Q',\theta}\,d\theta\,dx$$
$$\leqslant \sum_{k=1}^{n-j+1} \max\bigl\{\nu_P \colon P \in A_{j',k} \text{ for some } j' = j-1, j, j+1\bigr\}\bigl(|A_{j-1,k}| + |A_{j,k}| + |A_{j+1,k}|\bigr) \leqslant Cn4^{-2j}.$$

Estimate (7) is elementary when $r = \infty$. When $cn \leqslant r < \infty$, we exploit a relationship between circular Favard length and the area of the set of centers of the intersecting arcs, i.e., $(r + x)\,dx\,d\theta \approx r\,dx\,d\theta$ implies that $\nu_P \approx \frac{1}{r}|A|$, where A is the intersection of two annuli centered at q and q', both having inner radius $r - 4^{-n}$ and outer radius $r + 4^{-n}$. One can bound A by a rectangle and get the desired estimate by the Mean Value Theorem, for example. This concludes the proof of Theorem 2.1.

3. Sharpness of the $L \log\log L$ bound on the circular maximal operator

Let $c_m(z) := \{\zeta \colon |z - \zeta| = 4^{-m}\}$, and $Mf(z) := \sup_{m \geqslant 0} 4^m \int_{c_m(z)} |f(\zeta)||d\zeta|$. In [3], it was proved that $M : L(\log\log L) \to L^{1,\infty}$ is bounded, and then suggested that Favard length estimates could prove the sharpness. While this does not seem to be true, it still seems likely that a positive answer may be given by measuring the level sets of $f_{n,\theta,\sigma}$. Here and in [2], only the set $f_{n,\theta,\sigma} \geqslant 1$ was measured.

It is enough to show that for each $\varepsilon > 0$, $M : L(\log\log L)^{1-\varepsilon} \to L^{1,\infty}$ is *not* bounded, which follows if one can construct sets $E_n, |E_n| \ll 1$, such that $\sup_t t|\{z \colon M\chi_{E_n}(z) \geqslant t\}| \gg |E_n|(\log\log \frac{1}{|E_n|})^{1-\varepsilon}$.

The idea: $m < n$ will vary. Take a contraction $\widetilde{E_n}$ of $\mathcal{K}_n$ (by the factor 4^{-n}), and then take an $\varepsilon \approx 4^{-2n}$ neighborhood of this, called E_n. On a certain set of distance about 4^{-m} from E_n, there is a relatively large set of centers of circles of radius 4^{-m} which intersect $\widetilde{E_n}$, so that on this set, $M\chi_{E_n}$ is relatively large. Note that $|E_n| \approx 4^n \cdot 4^{-4n} = 4^{-3n}$, so that $\log\log \frac{1}{|E_n|} \approx \log n$.

Let $\mu_{n,m} := \{z \colon M\chi_{E_n} \geqslant 4^{m-2n}/(2\pi)\}$. Let $H_{n,m} := \{z \colon c_m(z) \cap \widetilde{E_n} \neq \emptyset\}$. Then $H_{n,m} \subset \mu_{n,m}$, $H_{n,m} \cap H_{n,m'} = \emptyset$ for $m \neq m'$, and $|H_{n,m}| \geqslant C4^{-m} Fav_\sigma(\widetilde{E_n}) \geqslant C\frac{\log n}{n}4^{-n-m}$.

Thus $|\bigcup_{m=0}^n \mu_{n,m}| \geqslant \sum_{m=0}^n |H_{n,m}| \geqslant \sum_{m=0}^n C\frac{\log n}{n}4^{-n-m}$. It would be nice if we could instead write the following for, say, $M = \alpha n$, for some constant $\alpha > 0$:

$$|\mu_{n,M}| \geqslant \sum_{m=0}^{M} C\frac{\log n}{n}4^{-n-M} \geqslant C\alpha \log n\, 4^{-n}4^{-M}, \tag{8}$$

because then

$$\frac{4^{M-2n}}{2\pi}\bigl|\bigl\{z \colon M\chi_{E_n} \geqslant 4^{M-2n}/(2\pi)\bigr\}\bigr| \geqslant C4^{-3n}\log n \geqslant C|E_n| \log\log\frac{1}{|E_n|} \gg |E_n|\left(\log\log\frac{1}{|E_n|}\right)^{1-\varepsilon}.$$

[2] Proof of Lemma 2.2: Write $z - w = \rho e^{i\theta}$, and let $\alpha := \arg(z-w) - \arg(T(z) - T(w))$.

$$\arg\bigl(T(z) - T(w)\bigr) = \arg\bigl((T - Id)(z) - (T - Id)(w) + (z - w)\bigr) = \arg\bigl(\lambda\rho e^{i\beta} + \rho e^{i\theta}\bigr)$$

for some $\lambda < \varepsilon, \beta \in [0, 2\pi]$. So $\arg(T(z) - T(w)) = \arg(\lambda e^{i\beta} + e^{i\theta})$. Then $|\alpha| \leqslant \hat{\alpha}$, where $\tan(\hat{\alpha}) = \frac{\varepsilon}{1-\varepsilon} \Rightarrow |\alpha| < 2\varepsilon$.

Let us state how one might get this. We can call by Q_j ($j = 1, \dots, 4^n$), the squares composing $\widetilde{E_n}$, and let $H_{n,m,M} := \{z\colon (\#j\colon c_m(z) \cap Q_j \neq \emptyset) \geqslant 4^{M-m}\}$. Then $H_{n,m,M} \subset \mu_{n,M}$.

Relation (8) would then follow if we had $|H_{n,m,M}| \geqslant C \frac{\log n}{n} 4^{-n-M}$. So we have this *strong conjecture*:

There exist $\alpha, C > 0$ such that for infinitely many n, $|\{(x,\theta) \in \mathbb{R} \times [0, 2\pi]\colon f_{n,\theta,\sigma}(x) \geqslant 4^m\}| \geqslant C \frac{\log n}{n} 4^{-m}$ for all $m \leqslant \alpha n$.

Alternately, a *weak conjecture*:

For all $\varepsilon > 0$, there exist $C > 0$ so that if

$$\nu(n) := \#\left\{m \leqslant n\colon \left|\left\{(x,\theta) \in \mathbb{R} \times [0, 2\pi]\colon f_{n,\theta,\sigma}(x) \geqslant 4^m\right\}\right| \geqslant C \frac{(\log n)^{1-\varepsilon}}{n} 4^{-m}\right\},$$

$$\text{then } \limsup_n \frac{\nu(n)}{n} (\log n)^{\varepsilon} > 0.$$

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