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A conditional least squares approach to *PGARCH* and *PARMA–PGARCH* time series estimation

L'estimateur des moindres carrés conditionnels dans les modèles PGARCH et PARMA–PGARCH

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ABSTRACT

In this Note, a conditional least squares (*CLS*) estimates for periodic GARCH (*PGARCH*) models with martingale difference centered squared innovations is developed. The approach is extended to the *PARMA–PGARCH* models. We establish the strong consistency and the asymptotic normality for our estimate.

RÉSUMÉ

Dans cette Note, on étudie l'estimateur des moindres carrés conditionnels (*CLS*) dans les modèles *GARCH* périodiques (*PGARCH*) dont le carré centré des innovations est une différence de martingale. Cette approche est étendue aux modèles *PARMA–PGARCH*. La consistance forte et la normalité asymptotique ont été établies.

Version française abrégée

Cette Note étudie la consistance forte et la normalité asymptotique de l'estimateur des moindres carrés conditionnels (*CLS*) dans les modèles *GARCH* périodiques (2) dont le carré centré des innovations est une différence de martingale. Cette approche est étendue aux modèles *PARMA–PGARCH* (4). En s'appuyant sur les versions multivariées (3) et (5), nous donnons des conditions (**A1**–**A9** ci dessous) de régularité assurant la stabilité et l'identifiabilité des modèles (2), (4) et sous lesquelles nous avons

Théorème 1. *Soit* $(\epsilon_t)_{t\in\mathbb{Z}}$ *solution de l'équation* (2), *alors*

1. *Sous les conditions* **A1**–**A4**, $\underline{\hat{\theta}}_n \to \underline{\theta}_0$ *presque sûrement quand* $n \to \infty$.
2. *Sous les conditions* **A1**–**A6**, $\sqrt{n}(\underline{\hat{\theta}}_n - \underline{\theta}_0)$ *converge en loi vers* $\mathcal{N}(\underline{O}, \mathcal{J}^{-1}\mathcal{I}\mathcal{J}^{-1})$ *quand* $n \to \infty$ *et où les matrices* $\mathcal{I}$ *et* $\mathcal{J}$ *sont définies dans le Théorème* 2.2.

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Théorème 2. *Soit* $(X_t)_{t\in\mathbb{Z}}$ *solution de l'equation* (4), *alors*

1. *Sous les conditions* **A2**–**A4** *et* **A7**–**A8**, $\underline{\hat{\pi}}_n \to \underline{\pi}_0$ *presque sûrement quand* $n \to \infty$.
2. *Sous les conditions* **A2**–**A9**, $\sqrt{n}(\underline{\hat{\pi}}_n - \underline{\pi}_0)$ *converge en loi vers* $\mathcal{N}\left(\binom{\underline{O}}{\underline{O}}, \begin{pmatrix} V_{11} & V_{12} \\ V_{21} & V_{22}\end{pmatrix}\right)$ *quand* $n \to \infty$ *et où les matrices* $(V_{i,j})_{1\leqslant i,j\leqslant 2}$ *sont définies dans le Théorème* 3.2.

Les théorèmes précédents sont obtenues en combinant des résultats de Francq et Zakoïan [5] et de Aknouche et Bibi [1]. Ces résultats sont appliqués à l'estimateur des moindres carrés conditionnels pour les modèles *GARCH* et *ARMA–PGARCH* périodiques solutions de (2) et de (4). Certains détails des preuves de cette Note figurent dans [3] et dans [1].

1. Introduction

In the process of attempting to model the conditional variance in financial time series $(\epsilon_n)_{n\in\mathbb{Z}}$ exhibiting structural changes, Bollerslev and Ghysels [4] have proposed a *GARCH*(p,q) model with time-varying coefficient which has the form of

$$\forall n \in \mathbb{Z}: \quad \epsilon_n = e_n\sqrt{h_n} \quad \text{and} \quad h_n = a_0(n) + \sum_{i=1}^{q} a_i(n)\epsilon_{n-i}^2 + \sum_{j=1}^{p} b_j(n)h_{n-j} \tag{1}$$

where $(e_n)_{n\in\mathbb{Z}}$ is a sequence of random variables (its characteristics are specified below), the coefficients $(a_i(n))_{0\leqslant i\leqslant q}$ and $(b_j(n))_{1\leqslant j\leqslant p}$ are positive except that $a_0(n) > 0$. The model (1) is called periodic *GARCH* (*PGARCH*) when the functions $(a_i(n))_{0\leqslant i\leqslant q}$ and $(b_j(n))_{1\leqslant j\leqslant p}$ are periodic in n with period $s > 0$, i.e., $a_i(n) = a_i(n+sk)$ and $b_j(n) = b_j(n+sk)$ for all integers $n, k \in \mathbb{Z}$. So, by setting $n = st + v$, $1 \leqslant v \leqslant s$, model (1) may be equivalently written as

$$\epsilon_{st+v} = e_{st+v}\sqrt{h_{st+v}} \quad \text{and} \quad h_{st+v} = a_0(v) + \sum_{i=1}^{q} a_i(v)\epsilon_{st+v-i}^2 + \sum_{j=1}^{p} b_j(v)h_{st+v-j}. \tag{2}$$

The *PGARCH* models are generally nonstationary but are stationary within each period, are becoming an appealing tool for investigating both volatility and distinct "seasonal" patterns and continue to gain a growing interest (see, e.g., Bibi and Lescheb [3] and the references therein for further discussion). Unfortunately, its probabilistic and statistical properties remain unexplored compared with respect to the other structures (for instance standard and Markovian switching *GARCH* models). The main reason is certainly, that the lack of stationarity and thus the ergodicity in such models result in enormous technical difficulties. Since the seminal paper by Pagano [7], with periodic coefficients, model (2) may be connected with multivariate model with time-invariant coefficient. More precisely $\underline{\epsilon}_t = (\epsilon_{st+1}, \dots, \epsilon_{st+s})'$ is a s-variate *GARCH*(p^*, q^*) model in the sense that

$$\underline{\epsilon}_t = \left\{\operatorname{diag}\{\underline{h}_t\}\right\}^{\frac{1}{2}}\underline{e}_t \quad \text{and} \quad \underline{h}_t = \underline{a}_0 + \sum_{i=0}^{q^*} A_i\underline{\epsilon}_{t-i}^2 + \sum_{j=0}^{p^*} B_j\underline{h}_{t-j} \tag{3}$$

where $\underline{\epsilon}_t^2 = (\epsilon_{st+1}^2, \dots, \epsilon_{st+s}^2)'$, $\underline{h}_t = (h_{st+1}, \dots, h_{st+s})'$ and $\underline{e}_t = (e_{st+1}, \dots, e_{st+s})'$. The model orders in (3) are $p^* = [\frac{p}{s}]$ and $q^* = [\frac{q}{s}]$ where $[x]$ denotes the smallest integer greater than or equal to x. The $s \times s$ matrices $(A_i)_{0\leqslant i\leqslant q^*}$ and $(B_i)_{0\leqslant i\leqslant p^*}$ are computed as follows (see Basawa and Lund [2]). A_0, B_0 have (i,j)th entries $(B_0)_{i,j} = b_{i-j}(i)\mathbb{I}_{\{i>j\}}$, $(A_0)_{i,j} = a_{i-j}(i)\mathbb{I}_{\{i>j\}}$ and $(B_m)_{i,j} = b_{ms+i-j}(i)$ for $1 \leqslant m \leqslant p^*$, $(A_m)_{i,j} = a_{ms+i-j}(i)$ for $1 \leqslant m \leqslant q^*$ and the intercept vector $\underline{a}_0 = (a_0(1), \dots, a_0(s))'$. Hence, the process $(\epsilon_n)_{n\in\mathbb{Z}}$ is said to be strictly periodically stationary (*SPS*) (resp. periodically ergodic) if $(\underline{\epsilon}_n)_{n\in\mathbb{Z}}$ is strictly stationary (resp. $(\underline{\epsilon}_n)_{n\in\mathbb{Z}}$ is ergodic).

In the sequel, $I_{(k)}$ denotes the identity matrix of order k, O (resp. $\underline{O}$) denotes the matrix (resp. vector) whose entries are zeros. The norm of a matrix $M = (m_{ij})$ is defined by $\|M\|$. This Note investigates the strong consistency and asymptotic normality of the conditional least squares (*CLS*) estimates for *PGARCH* and extends those asymptotic results to *PARMA–PGARCH* models. Our proofs closely follow those in Francq and Zakoïan [6] for independent and identically distributed innovations.

2. Least squares estimation for *PGARCH* models

Consider the *PGARCH* model (2) described with the vector of parameters $\underline{\theta} = (\underline{\theta}'(1), \dots, \underline{\theta}'(s))'$ where $\underline{\theta}(v) = (a_0(v), a_1(v), \dots, a_q(v), b_1(v), \dots, b_p(v))'$, $v = 1, \dots, s$. The vector $\underline{\theta}$ belongs to a parameter space $\Theta_{\underline{\theta}} := \{\underline{\theta}\colon \underline{\theta} \in (]0,\infty[\times [0,\infty[^{(p+q)})^s\}$. The orders p and q and the period s are supposed to be known, whereas the true parameter value $\underline{\theta}_0$ is unknown. Let $\{\epsilon_1, \epsilon_2, \ldots, \epsilon_N\}$ be a realization of length $N = sn$ to estimate the parameter $\underline{\theta}$. Conditionally on initial values $\epsilon_0, \epsilon_{-1}, \ldots, \epsilon_{1-q}$, $\hat{h}_0, \hat{h}_{-1}, \ldots, \hat{h}_{1-p}$ properly chosen, the *CLS* estimator of $\underline{\theta}$ is defined as any measurable solution $\underline{\hat{\theta}}_n$ of

$$\underline{\hat{\theta}}_n = Arg\min_{\underline{\theta}\in\Theta_{\underline{\theta}}} \widehat{Q}_n(\underline{\theta}), \quad \widehat{Q}_n(\underline{\theta}) := \frac{1}{n}\sum_{t=0}^{n-1} \hat{l}_t(\underline{\theta}), \quad \hat{l}_t(\underline{\theta}) := \frac{1}{s}\sum_{v=1}^{s} \hat{\eta}^2_{st+v}(\underline{\theta}) \quad \text{with } \hat{\eta}_{st+v}(\underline{\theta}) = \epsilon^2_{st+v} - \hat{h}_{st+v}(\underline{\theta})$$

where $\hat{h}_{st+v}(\theta)$ are defined recursively by $\hat{h}_{st+v}(\underline{\theta}) = a_0(v) + \sum_{i=1}^{q} a_i(v)\epsilon^2_{st+v-i} + \sum_{j=1}^{p} b_j(v)\hat{h}_{st+v-j}(\underline{\theta})$. For the strong consistency of $\underline{\hat{\theta}}_n$ we need the following regularity conditions. First define the local polynomials $\mathcal{A}_v(z) = \sum_{j=1}^{q} a_{0j}(v)z^j$, $\mathcal{B}_v(z) = 1 - \sum_{j=1}^{p} b_{0j}(v)z^j$ and assume that

A1. $\underline{\theta}_0 \in \Theta_{\underline{\theta}}$ and $\Theta_{\underline{\theta}}$ is compact.

A2. $(e_n)_{n\in\mathbb{Z}}$ is a sequence of strictly stationary and ergodic random variables satisfying almost surely $E\{e^2_n|\Im^{(\epsilon)}_{n-1}\} = 1$ where $\Im^{(\epsilon)}_n$ refers to the σ-field generated by $\{\epsilon_t, t \leqslant n\}$. Moreover, $(e^2_t)_{t\in\mathbb{Z}}$ has a non-degenerate distribution.

A3. The polynomial $\det(I_{(s)} - \sum_{j=0}^{\max(p^*+q^*)}(A_j + B_j)z^j)$ has its roots outside the unit circle and $E\{\epsilon^4_t\} < \infty$.

A4. For all $v \in \{1, \ldots, s\}$ and $\underline{\theta} \in \Theta_{\underline{\theta}}$, the local polynomials $\mathcal{A}_v(z)$ and $\mathcal{B}_v(z)$ have no common roots. Moreover, $\mathcal{A}_v(1) \neq 0$ and $a_{0q}(v) + b_{0p}(v) \neq 0$.

The first condition in **A3** ensures that Eq. (3) has a second order stationary, $\Im^{(e)}_n$-measurable, ergodic solution and $\det(I_{(s)} - \sum_{j=0}^{p^*} B_j z^j) \neq 0$ for all z such that $|z| \leqslant 1$. The last condition ensures that $h_t(\underline{\theta})$ has a causal solution of $\{\epsilon_t, \epsilon_{t-1}, \ldots\}$, i.e., $h_{st+v}(\underline{\theta}) = \alpha_0(v) + \sum_{j=1}^{\infty} \alpha_j(v)\epsilon^2_{st+v-j}$ for all $v \in \{1, \ldots, s\}$ in which the weights $\alpha_j(v)$ satisfy $\max_{1\leqslant v\leqslant s} \alpha_j(v) = O(\rho^j)$ with $\rho \in]0, 1[$.

Theorem 2.1. *Under* **A1**–**A4**, *almost surely* $\underline{\hat{\theta}}_n \to \underline{\theta}_0$ *as* $n \to \infty$.

In order to establish the asymptotic normality of *CLS–PGARCH*, let $\kappa_t := E\{e^4_t|\Im^{(\epsilon)}_{t-1}\}$ and consider the additional assumptions

A5. $\underline{\theta}_0 \in \Theta^o_{\underline{\theta}}$ where $\Theta^o_{\underline{\theta}}$ denotes the interior of $\Theta_{\underline{\theta}}$.

A6. $E\{e^4_t\} < \infty$ and $E\{\epsilon^8_t\} < \infty$.

Theorem 2.2. *Under* **A1**–**A6**, $\sqrt{n}(\underline{\hat{\theta}}_n - \underline{\theta}_0) \rightsquigarrow \mathcal{N}(\underline{O}, \mathcal{J}^{-1}\mathcal{I}\mathcal{J}^{-1})$ *where* $\mathcal{J}^{-1} := \operatorname{diag}\{\mathcal{J}_l^{-1}, l = 1, \ldots, s\}$, $\mathcal{I} := \operatorname{diag}\{\mathcal{I}_l, l = 1, \ldots, s\}$ *and each block matrix is given by*

$$\mathcal{J}_l := \sum_{v=1}^{s} E_{\underline{\theta}_0}\left\{\frac{\partial h_{st+v}(\underline{\theta})}{\partial\underline{\theta}(l)}\frac{\partial h_{st+v}(\underline{\theta})}{\partial\underline{\theta}(l)'}\right\}, \qquad \mathcal{I}_l := \sum_{v=1}^{s} E_{\underline{\theta}_0}\left\{(\kappa_{st+v} - 1)h^2_{st+v}(\underline{\theta})\frac{\partial h_{st+v}(\underline{\theta})}{\partial\underline{\theta}(l)}\frac{\partial h_{st+v}(\underline{\theta})}{\partial\underline{\theta}(l)'}\right\}.$$

The *CLS* estimator is not efficient due to the heteroscedasticity. To design a more efficient estimator of $\underline{\theta}$, we weight appropriately the nonlinear innovations $\hat{\eta}_t(\underline{\theta})$. Consider therefore $\hat{l}^{(\tau)}_t(\underline{\theta}) := \frac{1}{s}\sum_{v=1}^{s} \tau_{st+v}\hat{\eta}^2_{st+v}(\underline{\theta})$ where $\tau := (\tau_t)_t$ is a sequence of positive weights $\Im^{(\epsilon)}_{t-1}$ measurable. Hence it is easy to show that the weighted *CLS* is more efficient than the unweighted one when for instance $\tau_t = h_t^{-2}$. For this sequence of weights, the assumptions $E\{\epsilon^4_t\} < +\infty$ and $E\{\epsilon^8_t\} < +\infty$ may be replaced by $E\{|\epsilon|^{2\delta}_t\} < +\infty$ for some $\delta > 0$.

3. Estimation of *PARMA–PGARCH* processes

Consider a set of observations $\{X_1, \ldots, X_N; N = ns\}$ obtained from a centred *PARMA*(P, Q)*–PGARCH*(p, q) process $(X_t, t \in \mathbb{Z})$ satisfying

$$\begin{cases} X_t = \sum_{i=1}^{P} \phi_i(t)X_{t-i} + \epsilon_t - \sum_{j=1}^{Q} \varphi_j(t)\epsilon_{t-j} \\ \epsilon_t = \sqrt{h_t}e_t, h_t = a_0(t) + \sum_{i=1}^{q} a_i(t)\epsilon^2_{t-i} + \sum_{j=1}^{p} b_j(t)h_{t-j} \end{cases} \tag{4}$$

the coefficients $(\phi_i(t))_{1\leqslant i\leqslant P}$ and $(\varphi_j(t))_{1\leqslant j\leqslant Q}$ are periodic in t with period s. Assume that the process $(X_t, t \in \mathbb{Z})$ is described by a vector of parameters of interest $\underline{\pi} := (\underline{\beta}', \underline{\theta}')'$ where $\underline{\beta} = (\underline{\beta}'(1), \ldots, \underline{\beta}'(s))'$ with $\underline{\beta}(v) := (\phi_1(v), \ldots, \phi_P(v),$

$\varphi_1(v), \ldots, \varphi_Q(v))'$, $1 \leqslant v \leqslant s$ and the parameter space is $\Theta_{\underline{\pi}} \subset \Theta_{\underline{\beta}} \times \Theta_{\underline{\theta}}$ where $\Theta_{\underline{\beta}} := \mathbb{R}^{s(P+Q)}$. The orders P, Q, p and q and the period s are supposed to be known, unlike the true parameter value $\underline{\pi}_0 = (\underline{\beta}_0', \underline{\theta}_0')'$ is unknown. The corresponding vectorial version is

$$\begin{cases} \underline{X}_t = \sum_{i=0}^{P^*} \Phi_i \underline{X}_{t-i} + \underline{\epsilon}_t - \sum_{j=0}^{Q^*} \Psi_j \underline{\epsilon}_{t-j} \\ \underline{\epsilon}_t = \left\{\mathrm{diag}\{\underline{h}_t\}\right\}^{\frac{1}{2}} \underline{e}_t \quad \text{and} \quad \underline{h}_t = \underline{a}_0 + \sum_{i=0}^{q^*} A_i \underline{\epsilon}_{t-i}^2 + \sum_{j=0}^{p^*} B_j \underline{h}_{t-j} \end{cases} \tag{5}$$

where the matrices $(\Phi_i)_{0\leqslant i\leqslant P^*}$, $(\Psi_i)_{0\leqslant i\leqslant Q^*}$ and the orders P^*, Q^* may be computed as for *PGARCH*(p,q) (see Section 1). Conditionally on initial values $X_0, \ldots, X_{1-P-(q-Q)}$, $\widetilde{\epsilon}_{-(q-Q)}, \ldots, \widetilde{\epsilon}_{-1-q}$, $\widetilde{h}_0, \ldots, \widetilde{h}_{1-p}$ properly chosen (cf. Aknouche and Bibi [1]), the sequence of random vectors $\underline{\hat{\pi}}_n = (\underline{\hat{\beta}}_n', \underline{\hat{\theta}}_n')'$ is called two stages conditional least squares estimator if it satisfies almost surely

$$\underline{\hat{\beta}}_n = Arg \min_{\underline{\beta}\in\Theta_{\underline{\beta}}} \widehat{Q}_{1,n}(\underline{\beta}), \qquad \underline{\hat{\theta}}_n := Arg \min_{\underline{\theta}\in\Theta_{\underline{\theta}}} \widehat{Q}_{2,n}(\underline{\hat{\beta}}_n, \underline{\theta})$$

where $\widehat{Q}_{1,n}(\underline{\beta}) := \frac{1}{n}\sum_{t=0}^{n-1} \hat{l}_{1,t}(\underline{\beta})$ with $\hat{l}_{1,t}(\underline{\beta}) := \frac{1}{s}\sum_{v=1}^{s} \hat{\epsilon}_{st+v}^2(\underline{\beta})$ and where $\widehat{Q}_{2,n}(\underline{\pi}) := \frac{1}{n}\sum_{t=0}^{n-1} \hat{l}_{2,t}(\underline{\pi})$ with $\hat{l}_{2,t}(\underline{\pi}) := \frac{1}{s}\sum_{v=1}^{s} \hat{\eta}_{st+v}^2(\underline{\pi})$. For $v = 1, \ldots, s$, consider the local polynomials $\Phi_v(z) = 1 - \sum_{i=1}^{P} \phi_{0i}(v) z^i$, $\Psi_v(z) = 1 - \sum_{i=1}^{Q} \varphi_{0i}(v) z^i$ and we introduce the following conditions

A7. $\underline{\pi}_0 \in \Theta_{\underline{\pi}}$ and $\Theta_{\underline{\pi}}$ is compact.
A8. The polynomials $\det(I_{(s)} - \sum_{j=0}^{P^*} \Phi_j z^j)$ and $\det(I_{(s)} - \sum_{j=0}^{Q^*} \Psi_j z^j)$ have their roots outside the unit circle with $\phi_{0P}(v) \neq 0$ or $\varphi_{0Q}(v) \neq 0$ for all $v = 1, \ldots, s$.
A9. $\underline{\pi}_0$ is in the interior of $\Theta_{\underline{\pi}}$.

Note here that under **A8**, $(X_t)_{t\in\mathbb{Z}}$ and $(\epsilon_t)_{t\in\mathbb{Z}}$ can be related through the infinite order moving average and autoregressive expansions $X_{st+v} = \sum_{i=0}^{\infty} \alpha_i(v)\epsilon_{st+v-i}$ and $\epsilon_{st+v} = \sum_{i=0}^{\infty} \beta_i(v) X_{st+v-i}$ in which the weights $\alpha_i(v)$ and $\beta_i(v)$ satisfy $\sup_{1\leqslant v\leqslant s} |\alpha_i(v)| = O(\rho^i)$ and $\sup_{1\leqslant v\leqslant s} |\beta_i(v)| = O(\rho^i)$ with $0 < \rho < 1$. So, the local polynomials $\Phi_v(z)$ and $\Psi_v(z)$ have all their roots outside the unit circle and have no common root.

Theorem 3.1. *Let $(X_t)_{t\in\mathbb{Z}}$ be a PARMA–PGARCH process satisfying* (4). *Then under* **A2–A4** *and* **A7–A8**, *almost surely $\underline{\hat{\pi}}_n \to \underline{\pi}_0$ as $n \to \infty$.*

The limit distribution of $\underline{\hat{\pi}}_n$ is given in the following theorem.

Theorem 3.2. *Let $(X_t)_{t\in\mathbb{Z}}$ be a PARMA–PGARCH process satisfying* (4). *Then under* **A2–A9** *we have*

$$\sqrt{n}(\underline{\hat{\pi}}_n - \underline{\pi}_0) \rightsquigarrow \mathcal{N}\left(\begin{pmatrix} \underline{O} \\ \underline{O} \end{pmatrix}, \begin{pmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{pmatrix}\right)$$

where $V_{11} = J_{11}^{-1} I_{11} J_{11}^{-1}$, $V_{12} = V_{21}' = J_{22}^{-1}(I_{21} + J_{21} J_{11}^{-1} I_{11}) J_{11}^{-1}$, $V_{22} = J_{22}^{-1}(I_{22} + J_{21} J_{11}^{-1} I_{11} J_{11}^{-1} J_{12} - I_{21} J_{11}^{-1} J_{12} - J_{21} J_{11}^{-1} I_{12}) \times J_{22}^{-1}$ *with*

$$I_{11} = \lim_{n\to\infty} Var_{\underline{\beta}_0}\left\{\sqrt{n}\frac{\partial}{\partial\underline{\beta}}\widehat{Q}_{1,n}(\underline{\beta})\right\}, \qquad I_{22} = \lim_{n\to\infty} Var_{\underline{\pi}_0}\left\{\sqrt{n}\frac{\partial}{\partial\underline{\theta}}\widehat{Q}_{2,n}(\underline{\pi})\right\},$$

$$I_{12} = \lim_{n\to\infty} E_{\underline{\pi}_0}\left\{n\frac{\partial}{\partial\underline{\beta}}\widehat{Q}_{1,n}(\underline{\beta})\frac{\partial}{\partial\underline{\theta}'}\widehat{Q}_{2,n}(\underline{\pi})\right\}, \qquad I_{21} = I_{12}',$$

$$J_{11} = \lim_{n\to\infty} Var_{\underline{\beta}_0}\left\{\frac{\partial^2}{\partial\underline{\beta}\partial\underline{\beta}'}\widehat{Q}_{1,n}(\underline{\beta})\right\}, \qquad J_{22} = \lim_{n\to\infty} Var_{\underline{\beta}_0}\left\{\frac{\partial^2}{\partial\underline{\theta}\partial\underline{\theta}'}\widehat{Q}_{2,n}(\underline{\pi})\right\},$$

$$J_{12} = \lim_{n\to\infty} Var_{\underline{\beta}_0}\left\{\frac{\partial^2}{\partial\underline{\beta}\partial\underline{\theta}'}\widehat{Q}_{2,n}(\underline{\pi})\right\}, \qquad J_{21} = J_{12}'.$$

4. Proofs

The proofs of Theorems 2.1 and 3.1 are by now standard and follow from similar arguments used in showing the strong consistency of the *QMLE–PGARCH* and *QMLE–PARMA–PGARCH* models (cf. Aknouche and Bibi [1] and Bibi and Lescheb [3]) and hence, we do not detail the proofs. Thus, we give only a sketch of proof for the asymptotic normality and refer to Aknouche and Bibi [1], Francq and Zakoïan [6,5] or Bibi and Lescheb [3] for further details. Because $(\hat{l}_t(\underline{\theta}))_{t\in\mathbb{Z}}$ (resp. $(\hat{l}_{1,t}(\underline{\beta}))_{t\in\mathbb{Z}}$, $(\hat{l}_{2,t}(\underline{\pi}))_{t\in\mathbb{Z}}$) is not an *SPS* process due to the presence of initial values, we shall replace it by its *SPS* version $(l_t(\underline{\theta}))_{t\in\mathbb{Z}}$ (resp. $(l_{1,t}(\underline{\beta}))_{t\in\mathbb{Z}}$, $(l_{2,t}(\underline{\pi}))_{t\in\mathbb{Z}}$) in which no constraints on the initial values were imposed.

4.1. Proof of Theorem 2.2 [Asymptotic normality of CLS–PGARCH]

Using Taylor-series expansion around $\underline{\theta}_0$, we obtain

$$\underline{O} = \frac{1}{\sqrt{n}}\sum_{t=0}^{n-1}\frac{\partial l_t(\underline{\hat{\theta}}_n)}{\partial\underline{\theta}} = \frac{1}{\sqrt{n}}\sum_{t=0}^{n-1}\frac{\partial l_t(\underline{\theta}_0)}{\partial\underline{\theta}} + \left(\frac{1}{n}\sum_{t=0}^{n-1}\frac{\partial^2 l_t(\underline{\theta}_*)}{\partial\underline{\theta}\partial\underline{\theta}'}\right)\sqrt{n}(\underline{\hat{\theta}}_n-\underline{\theta}_0)$$

where $\underline{\theta}_*$ is between $\underline{\hat{\theta}}_n$ and $\underline{\theta}_0$. Thus we need to show that $\frac{1}{\sqrt{n}}\sum_{t=0}^{n-1}\frac{\partial l_t(\underline{\theta}_0)}{\partial\underline{\theta}} \rightsquigarrow \mathcal{N}(\underline{O}, \frac{4}{s^2}\mathcal{I}(\underline{\theta}_0))$ and $\frac{1}{n}\sum_{t=0}^{n-1}\frac{\partial^2 l_t(\underline{\theta}_*)}{\partial\underline{\theta}\partial\underline{\theta}'} \xrightarrow{P} \frac{2}{s}\mathcal{J}(\underline{\theta}_0)$ and hence the result follows from Slutsky's theorem and the following intermediate results grouped in the following lemma.

Lemma 4.1. *Under* **A1–A6** *we have*

1. $E_{\underline{\theta}_0}\{\|\frac{\partial l_t(\underline{\theta})}{\partial\underline{\theta}}\frac{\partial l_t(\underline{\theta})}{\partial\underline{\theta}'}\|\} < +\infty$, $E_{\underline{\theta}_0}\{\|\frac{\partial^2 l_t(\underline{\theta})}{\partial\underline{\theta}\partial\underline{\theta}'}\|\} < +\infty$ *and* $E_{\underline{\theta}_0}\{\sup_{\underline{\theta}\in\vartheta(\underline{\theta}_0)}|\frac{\partial^3 l_t(\underline{\theta})}{\partial\theta_i\partial\theta_j\partial\theta_k}|\} < +\infty$ *for some neighborhood* $\vartheta(\underline{\theta}_0)$ *of* $\underline{\theta}_0$ *and for all* $i, j, k \in \{1,\dots,s(p+q+1)\}$.
2. $\|\frac{1}{\sqrt{n}}\sum_{t=0}^{n-1}\{\frac{\partial l_t(\underline{\theta})}{\partial\underline{\theta}} - \frac{\partial\hat{l}_t(\underline{\theta})}{\partial\underline{\theta}}\}\|$ *and* $\sup_{\underline{\theta}\in\vartheta(\underline{\theta}_0)}\|\frac{1}{n}\sum_{t=0}^{n-1}\{\frac{\partial^2 l_t(\underline{\theta})}{\partial\underline{\theta}\partial\underline{\theta}'} - \frac{\partial^2\hat{l}_t(\underline{\theta})}{\partial\underline{\theta}\partial\underline{\theta}'}\}\|$ *converges in probability to* 0 *as* $n\to\infty$.
3. $Var_{\underline{\theta}_0}\{\frac{\partial l_t(\underline{\theta})}{\partial\underline{\theta}}\} = \frac{4}{s^2}\mathcal{I}(\underline{\theta}_0)$.
4. $\frac{1}{\sqrt{n}}\sum_{t=0}^{n-1}\frac{\partial l_t(\underline{\theta}_0)}{\partial\underline{\theta}}$ *converges in distribution to* $\mathcal{N}(\underline{O}, \frac{4}{s^2}\mathcal{I}(\underline{\theta}_0))$.
5. $\frac{1}{n}\sum_{t=0}^{n-1}\frac{\partial^2 l_t(\underline{\theta}_*)}{\partial\underline{\theta}\partial\underline{\theta}'}$ *converges in probability to* $\frac{2}{s}\mathcal{J}(\underline{\theta}_0)$ *and* $\mathcal{J}(\underline{\theta}_0)$ *is non-singular matrix.*

Proof. The proof follows from standard arguments (cf. Aknouche and Bibi [1] and Francq and Zakoïan [6]). □

4.2. Proof of Theorem 3.2 [Asymptotic normality of CLS–PARMA–PGARCH]

The proof rests classically on the Taylor-series expansion around the true parameters values

$$\underline{O} = \frac{1}{\sqrt{n}}\sum_{t=0}^{n-1}\frac{\partial l_{1,t}(\underline{\hat{\beta}}_n)}{\partial\underline{\beta}} = \frac{1}{\sqrt{n}}\sum_{t=0}^{n-1}\frac{\partial l_{1,t}(\underline{\beta}_0)}{\partial\underline{\beta}} + \left(\frac{1}{n}\sum_{t=0}^{n-1}\frac{\partial^2 l_{1,t}(\underline{\beta}_*)}{\partial\underline{\beta}\partial\underline{\beta}'}\right)\sqrt{n}(\underline{\hat{\beta}}_n-\underline{\beta}_0)$$

$$\begin{aligned}\underline{O} &= \frac{1}{\sqrt{n}}\sum_{t=0}^{n-1}\frac{\partial l_{2,t}(\underline{\hat{\beta}}_n,\underline{\hat{\theta}}_n)}{\partial\underline{\theta}} = \frac{1}{\sqrt{n}}\sum_{t=0}^{n-1}\frac{\partial l_{2,t}(\underline{\hat{\beta}}_n,\underline{\theta}_0)}{\partial\underline{\theta}} + \left(\frac{1}{n}\sum_{t=0}^{n-1}\frac{\partial^2 l_{2,t}(\underline{\hat{\beta}}_n,\underline{\theta}_*)}{\partial\underline{\theta}\partial\underline{\theta}'}\right)\sqrt{n}(\underline{\hat{\theta}}_n-\underline{\theta}_0)\\ &= \frac{1}{\sqrt{n}}\sum_{t=0}^{n-1}\frac{\partial l_{2,t}(\underline{\beta}_0,\underline{\theta}_0)}{\partial\underline{\theta}} + \left(\frac{1}{n}\sum_{t=0}^{n-1}\frac{\partial^2 l_{2,t}(\underline{\beta}_{**},\underline{\theta}_0)}{\partial\underline{\theta}\partial\underline{\beta}'}\right)\sqrt{n}(\underline{\hat{\beta}}_n-\underline{\beta}_0) + \left(\frac{1}{n}\sum_{t=0}^{n-1}\frac{\partial^2 l_{2,t}(\underline{\hat{\beta}}_n,\underline{\theta}_*)}{\partial\underline{\theta}\partial\underline{\theta}'}\right)\sqrt{n}(\underline{\hat{\theta}}_n-\underline{\theta}_0),\end{aligned}$$

where $\underline{\beta}_*$'s (resp. $\underline{\beta}_{**}$, $\underline{\theta}_*$, $\underline{\pi}_*$) are between $\underline{\hat{\beta}}_n$ and $\underline{\beta}_0$ (resp. $\underline{\hat{\beta}}_n$ and $\underline{\beta}_0$, $\underline{\hat{\theta}}_n$ and $\underline{\theta}_0$ and between $\underline{\hat{\pi}}_n$ and $\underline{\pi}_0$). The above equations lead to

$$-\frac{1}{\sqrt{n}}\sum_{t=0}^{n-1}\frac{\partial \underline{l}_t(\underline{\pi}_0)}{\partial\underline{\pi}} = \frac{1}{n}\sum_{t=0}^{n-1}\frac{\partial^2 \underline{l}_t(\underline{\pi}_*)}{\partial\underline{\pi}\partial\underline{\pi}'}\sqrt{n}(\underline{\hat{\pi}}_n-\underline{\pi}_0)$$

where $\frac{\partial \underline{l}_t(\underline{\pi}_0)}{\partial\underline{\pi}} := (\frac{\partial l_{1,t}(\underline{\beta}_0)}{\partial\underline{\beta}'}, \frac{\partial l_{2,t}(\underline{\pi}_0)}{\partial\underline{\theta}'})'$. Thus we need to show that

$$\frac{1}{\sqrt{n}}\sum_{t=0}^{n-1}\frac{\partial \underline{l}_t(\underline{\pi}_0)}{\partial\underline{\pi}} \rightsquigarrow \mathcal{N}(\underline{O}, I) \quad\text{and}\quad \frac{1}{n}\sum_{t=0}^{n-1}\frac{\partial^2 \underline{l}_t(\underline{\pi}_*)}{\partial\underline{\pi}\partial\underline{\pi}'} \to J \quad\text{in probability with } I = \begin{pmatrix} I_{11} & I_{12}\\ I_{21} & I_{22}\end{pmatrix},\ J = \begin{pmatrix} J_{11} & O\\ J_{21} & J_{22}\end{pmatrix}$$

where the sub-matrices $(I_{ij})_{1\leqslant i,j\leqslant 2}$ and $(J_{ij})_{1\leqslant i,j\leqslant 2}$ are given in Theorem 3.2. For this purpose, we show an analogue Lemma 4.1 for $\frac{\partial l_t(\underline{\pi}_0)}{\partial \underline{\pi}}$.

Lemma 4.2. *If Assumptions* **A2**–**A9** *hold, then*

1. *For any* $\underline{\pi} \in \Theta$, *the random vectors* $\frac{\partial}{\partial \underline{\beta}} l_{1,n}(\underline{\beta})$ *and* $\frac{\partial}{\partial \underline{\theta}} l_{2,n}(\underline{\pi})$ *exist and belong to* $\mathbb{L}_2$.
2. $E_{\underline{\pi}_0}\{\|\frac{\partial l_t(\underline{\pi})}{\partial \underline{\pi}}\frac{\partial l_t(\underline{\pi})}{\partial \underline{\pi}'}\|\} < +\infty$, $E_{\underline{\pi}_0}\{\|\frac{\partial^2 l_t(\underline{\pi})}{\partial \underline{\pi}\partial \underline{\pi}'}\|\} < +\infty$ *and* $E_{\underline{\pi}_0}\{\sup_{\underline{\theta}\in\vartheta(\underline{\theta}_0)}|\frac{\partial^3 l_t(\underline{\pi})}{\partial \pi_i \partial \pi_j \partial \pi_k}|\} < +\infty$ *for some neighborhood* $\vartheta(\underline{\pi}_0)$ *of* $\underline{\pi}_0$ *and for all* $i, j, k \in \{1, \ldots, s(p+q+P+Q+1)\}$.
3. $\|\frac{1}{\sqrt{n}}\sum_{t=0}^{n-1}\{\frac{\partial l_t(\underline{\pi})}{\partial \underline{\pi}} - \frac{\partial \hat{l}_t(\underline{\pi})}{\partial \underline{\pi}}\}\|$ *and* $\sup_{\underline{\pi}\in\vartheta(\underline{\pi}_0)}\|\frac{1}{n}\sum_{t=0}^{n-1}\{\frac{\partial_t^2 l(\underline{\pi})}{\partial \underline{\pi}\partial \underline{\pi}'} - \frac{\partial^2 \hat{l}_t(\underline{\pi})}{\partial \underline{\pi}\partial \underline{\pi}'}\}\|$ *converges in probability to* 0 *as* $n \to \infty$.
4. $\frac{1}{\sqrt{n}}\sum_{t=0}^{n-1}\frac{\partial l_t(\underline{\pi}_0)}{\partial \underline{\pi}}$ *converges in distribution to* $\mathcal{N}(\underline{O}, I(\underline{\pi}_0))$ *where the sub-matrices* I_{11}, I_{12} *and* I_{22} *exist and are strictly positive definite.*
5. $\frac{1}{n}\sum_{t=0}^{n-1}\frac{\partial_t^2 l(\underline{\pi}_*)}{\partial \underline{\pi}\partial \underline{\pi}'}$ *converges in probability to* $J(\underline{\pi}_0)$ *and* $(J_{ii}(\underline{\pi}_0))_{1\leqslant i\leqslant 2}$ *are non-singular matrices.*

Proof.

1. Note that under Assumptions **A6** and **A8** $E\{X_t^4\} < +\infty$. By the Cauchy–Schwartz inequality we can see that $\frac{\partial \epsilon_{st+v}^2(\underline{\beta})}{\partial \underline{\beta}} = 2\epsilon_{st+v}(\underline{\beta})\frac{\partial \epsilon_{st+v}(\underline{\beta})}{\partial \underline{\beta}}$ and $\frac{\partial \eta_{st+v}^2(\underline{\pi})}{\partial \underline{\theta}} = 2\eta_{st+v}(\underline{\pi})\frac{\partial \eta_{st+v}(\underline{\pi})}{\partial \underline{\theta}}$ belong to $\mathbb{L}_2$.
2. The statements in Assertions 2 and 3 follow similarly as proving Lemma 4.1.
3. By Assumptions **A6** and **A8**, we have $\Im_t^{(\epsilon)} = \Im_t^{(X)}$, $E_{\underline{\beta}_0}\{\frac{\partial l_{1,t}(\underline{\beta}_0)}{\partial \underline{\beta}}|\Im_{t-1}^{(X)}\} = \underline{O}$, $E_{\underline{\pi}_0}\{\frac{\partial l_{2,t}(\underline{\pi}_0)}{\partial \underline{\theta}}|\Im_{t-1}^{(X)}\} = \underline{O}$ and $Var_{\underline{\beta}_0}\{\frac{\partial l_{1,t}(\underline{\beta}_0)}{\partial \underline{\beta}}\}$ and $Var_{\underline{\pi}_0}\{\frac{\partial l_{2,t}(\underline{\pi}_0)}{\partial \underline{\theta}}\}$ exist and not singular matrices. Hence, for any $(\underline{\lambda}', \underline{\mu}')' \in \mathbb{R}^{s(P+Q)} \times \mathbb{R}^{s(p+q+1)}$, the sequence $\{(\underline{\lambda}', \underline{\mu}')\frac{\partial l_t(\underline{\pi})}{\partial \underline{\pi}}, \Im_t^{(X)}\}_t$ is a square integrable *SPS* martingale difference. The central limit theorem and the Wold–Cramèr device allow to derive the asymptotic normality result.
4. The convergence follows from the almost surely convergence of $\underline{\pi}_*$ to $\underline{\pi}_0$, an application of the ergodic theorem for $\frac{1}{n}\sum_{t=0}^{n-1}\frac{\partial^2 l_t(\underline{\pi}_*)}{\partial \underline{\pi}\partial \underline{\pi}'}$ and the fact that almost surely as $n \to \infty$

$$\left\|\frac{1}{n}\sum_{t=0}^{n-1}\left(\frac{\partial^2 l_{1,t}(\underline{\beta}_*)}{\partial \underline{\beta}\partial \underline{\beta}'} - \frac{\partial^2 l_{1,t}(\underline{\beta}_0)}{\partial \underline{\beta}\partial \underline{\beta}'}\right)\right\| \to 0, \qquad \left\|\frac{1}{n}\sum_{t=0}^{n-1}\left(\frac{\partial^2 l_{2,t}(\underline{\beta}_{**}, \underline{\theta}_0)}{\partial \underline{\theta}\partial \underline{\beta}'} - \frac{\partial^2 l_{2,t}(\underline{\pi}_0)}{\partial \underline{\theta}\partial \underline{\beta}'}\right)\right\| \to 0,$$

$$\left\|\frac{1}{n}\sum_{t=0}^{n-1}\left(\frac{\partial^2 l_{2,t}(\underline{\hat{\beta}}_n, \underline{\theta}_*)}{\partial \underline{\theta}\partial \underline{\theta}'} - \frac{\partial^2 l_{2,t}(\underline{\pi}_0)}{\partial \underline{\theta}\partial \underline{\theta}'}\right)\right\| \to 0. \qquad \square$$

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