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An *abc* theorem on the disk ☆

Un théorème du type abc sur le disque

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ABSTRACT

We extend the classical *abc* theorem for polynomials (also known as Mason's, or Mason–Stothers', theorem) to general analytic functions on the disk.

RÉSUMÉ

On généralise le « théorème *abc* » sur les polynômes (alias le théorème de Mason–Stothers) au cas des fonctions analytiques arbitraires sur le disque.

1. Introduction and main result

Given a polynomial p (in one complex variable), write $\deg p$ for the degree of p and $\widetilde{N}(p) = \widetilde{N}_{\mathbb{C}}(p)$ for the number of its distinct zeros in $\mathbb{C}$. The so-called *abc* theorem, often referred to as Mason's theorem (but essentially due to Stothers [8]), reads as follows:

Theorem A. *Suppose a, b and c are polynomials, not all constants, having no common zeros and satisfying $a + b = c$. Then*

$$\max\{\deg a, \deg b, \deg c\} \leqslant \widetilde{N}(abc) - 1. \tag{1}$$

Various approaches to and consequences of Theorem A are discussed in [4–7]. One impressive – and immediate – application is a simple proof of Fermat's Last Theorem for polynomials, saying that there are no nontrivial polynomial solutions to the equation $P^n + Q^n = R^n$ when $n \geqslant 3$. Besides, it was Theorem A that led (via the classical analogy between polynomials and integers) to the famous *abc conjecture* in number theory; see [4,6].

In this Note, we present an *abc* type theorem that applies to a much more general situation. Namely, we replace the polynomial equation $a + b = c$ by

$$f_0 + \cdots + f_n = f_{n+1}, \tag{2}$$

where the f_j's are analytic functions on the (closed) disk $\mathbb{D} \cup \mathbb{T}$. Here and below, we write $\mathbb{D}$ for the unit disk $\{z \in \mathbb{C}\colon |z| < 1\}$ and $\mathbb{T}$ for its boundary, $\partial\mathbb{D}$. The functions are thus assumed to be analytic in some open neighborhood of $\mathbb{D} \cup \mathbb{T}$.

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With each f_j we associate the (finite) Blaschke product B_j built from the function's zeros. This means that B_j is given by

$$z \mapsto \prod_{k=1}^{s}\left(\frac{z-a_k}{1-\bar{a}_k z}\right)^{m_k}, \tag{3}$$

where $a_k = a_k^{(j)}$ $(1 \leqslant k \leqslant s = s_j)$ are the distinct zeros of f_j in $\mathbb{D}$, and $m_k = m_k^{(j)}$ are their respective multiplicities. Further, let $\mathbf{B}$ denote the *least common multiple* of the Blaschke products $B_0, \dots, B_{n+1}$ (defined in the natural way), and put

$$\mathcal{B} := \operatorname{rad}(B_0 B_1 \dots B_{n+1}).$$

Here, we use the notation $\operatorname{rad}(B)$ for the *radical* of a Blaschke product B; this is, by definition, the Blaschke product that arises when the zeros of B are all converted into simple ones. In other words, given a Blaschke product of the form (3), its radical is obtained by replacing each m_k with 1.

Finally, we write $W = W(f_0, \dots, f_n)$ for the *Wronskian* of the (analytic) functions $f_0, \dots, f_n$, so that

$$W := \begin{vmatrix} f_0 & f_1 & \dots & f_n \\ f_0' & f_1' & \dots & f_n' \\ \dots & \dots & \dots & \dots \\ f_0^{(n)} & f_1^{(n)} & \dots & f_n^{(n)} \end{vmatrix}. \tag{4}$$

We then introduce the quantities

$$\kappa = \kappa(W) := \|W'\|_{L^1(\mathbb{T})}\|1/W\|_{L^\infty(\mathbb{T})},$$
$$\lambda = \lambda(W) := \|W'\|_{L^2(\mathbb{D})}\|1/W\|_{L^\infty(\mathbb{T})}$$

and

$$\mu = \mu(W) := \|W\|_{L^\infty(\mathbb{T})}\|1/W\|_{L^\infty(\mathbb{T})}.$$

(It is understood that $\mathbb{D}$ is endowed with the normalized area measure $\mathrm{d}A$, while $\mathbb{T}$ is endowed with the normalized arclength measure $\mathrm{d}m$.) The three quantities are finite, provided that W has no zeros on $\mathbb{T}$.

Theorem 1.1. *Suppose f_j $(j = 0, 1, \dots, n+1)$ are analytic functions on $\mathbb{D} \cup \mathbb{T}$, related by* (2) *and such that the Wronskian* (4) *vanishes nowhere on $\mathbb{T}$. Then*

$$N_{\mathbb{D}}(\mathbf{B}) \leqslant \kappa + n\mu N_{\mathbb{D}}(\mathcal{B}) \tag{5}$$

and

$$N_{\mathbb{D}}(\mathbf{B}) \leqslant \lambda^2 + n\mu^2 N_{\mathbb{D}}(\mathcal{B}), \tag{6}$$

where $N_{\mathbb{D}}(\cdot)$ denotes the number of the function's zeros in $\mathbb{D}$, counting multiplicities.

2. Discussion

(i) Neither of the inequalities (5) and (6) implies the other. However, (5) offers a smaller factor in front of $N_{\mathbb{D}}(\mathcal{B})$, since $\mu \geqslant 1$.

(ii) Both inequalities (5) and (6) are sharp. Consider, as an example, the functions $f_0(z) = 1$ and $f_j(z) = \varepsilon z^j/j!$ $(j = 1, \dots, n)$ with a suitable $\varepsilon > 0$; then define f_{n+1} by (2). If ε is small enough, then f_{n+1} is zero-free on $\mathbb{D}$. The Blaschke products that arise are $B_0(z) = B_{n+1}(z) = 1$ and $B_j(z) = z^j$ for $1 \leqslant j \leqslant n$, whence $\mathbf{B}(z) = z^n$ and $\mathcal{B}(z) = z$. Also, one easily checks that $W = \varepsilon^n (= \text{const})$, which implies $\kappa = \lambda = 0$ and $\mu = 1$. Consequently, equality holds in both (5) and (6).

To get an example where equality holds with nonzero κ and λ, take $f_0, \dots, f_{n-1}$ as above, then put $f_n(z) = \varepsilon z^m/m!$ for some integer $m > n$, and again define f_{n+1} by (2).

(iii) Any of the two estimates, (5) or (6), implies Theorem A. Let us explain how to derive (1) from (5). Given a, b and c as in Theorem A, write d for the left-hand side of (1). At least two of the polynomials, say a and b, must then be of degree d. Set $d_1 := \deg c$. Further, let R be a large positive number, ensuring that the disk $R\mathbb{D} = \{z\colon |z| < R\}$ contains the zeros of the three polynomials. Next, we adjust Theorem 1.1 to the disk $R\mathbb{D}$ in place of $\mathbb{D}$ (by rescaling) and apply it with $n = 1$, putting $f_0 = a$, $f_1 = b$ and $f_2 = c$. Since a, b and c have no common zeros, the least common multiple $\mathbf{B}$ coincides with the product of the three Blaschke products (the ones built from the three polynomials, regarded as functions on $R\mathbb{D}$); a similar remark applies to $\mathcal{B}$, once the three Blaschke products are replaced by their radicals. The left-hand side of the rescaled version of (5) is $N_{R\mathbb{D}}(\mathbf{B})$, or equivalently $N_{R\mathbb{D}}(abc)$, which reduces to the sum of the three degrees, i.e., to $2d + d_1$. Making a suitable replacement for $N_{R\mathbb{D}}(\mathcal{B})$ on the right-hand side, we obtain

$$2d + d_1 \leqslant \kappa_R + \mu_R \widetilde{N}(abc), \tag{7}$$

where κ_R and μ_R are the rescaled versions of κ and μ:

$$\kappa_R = \left(R \int_{\mathbb{T}} \left| W'(R\zeta) \right| \mathrm{d}m(\zeta) \right) \cdot \left(\min_{R\mathbb{T}} |W| \right)^{-1} \quad \text{and} \quad \mu_R = \left(\max_{R\mathbb{T}} |W| \right) \cdot \left(\min_{R\mathbb{T}} |W| \right)^{-1}.$$

The desired estimate (1) will now follow from (7) upon passing to the limit as $R \to \infty$. It suffices to notice that the Wronskian $W = W(a,b) = W(a,c)$ is a polynomial whose degree, say l, does not exceed $d + d_1 - 1$. It is the leading term, $\mathrm{const} \cdot z^l$, that determines the asymptotic behavior of κ_R and μ_R, so the limits of the two quantities are l and 1, respectively. Substituting these into (7) and using the bound $l \leqslant d + d_1 - 1$ yields (1).

(iv) Several extensions and refinements of Theorem 1.1 are available. In particular, the f_j's need not be analytic on the *closed* disk; we may assume instead that they are analytic on $\mathbb{D}$ and appropriately smooth up to $\mathbb{T}$. The functions may then have *infinitely many zeros* in $\mathbb{D}$, a complication that calls for a new approach. Finally, the disk $\mathbb{D}$ can be replaced by a general (reasonably nice) domain – say, by a bounded simply connected domain $\Omega \subset \mathbb{C}$ with $\partial\Omega$ a rectifiable Jordan curve. Some of these matters are treated in [3].

3. Proof of Theorem 1.1 (sketch)

We shall only outline the proof of (6). The first step consists in verifying that $\mathbf{B}$ divides $W\mathcal{B}^n$, in the sense that $W\mathcal{B}^n/\mathbf{B}$ is analytic on $\mathbb{D}$ (and in fact on $\mathbb{D} \cup \mathbb{T}$). This algebraic step is fairly elementary and can be accomplished by expanding the determinant along the appropriate column, while keeping track of the zeros; see [3] for details.

Now we know that $W\mathcal{B}^n = F\mathbf{B}$, with F analytic. Therefore,

$$\left\| \left(W\mathcal{B}^n \right)' \right\|^2_{L^2(\mathbb{D})} = \left\| (F\mathbf{B})' \right\|^2_{L^2(\mathbb{D})}, \tag{8}$$

and we proceed by estimating the two *Dirichlet integrals* that arise. Writing LHS (resp., RHS) for the left-hand (resp., right-hand) side of (8), we have

$$\mathrm{RHS} = \int_{\mathbb{D}} \left| (F\mathbf{B})' \right|^2 \mathrm{d}A = \int_{\mathbb{D}} \left| F' \right|^2 \mathrm{d}A + \int_{\mathbb{T}} |F|^2 \left| \mathbf{B}' \right| \mathrm{d}m.$$

The latter identity can be either deduced from Green's formula or readily borrowed from [1]; see also [2]. Consequently,

$$\mathrm{RHS} \geqslant \int_{\mathbb{T}} |F|^2 \left| \mathbf{B}' \right| \mathrm{d}m = \int_{\mathbb{T}} |W|^2 \left| \mathbf{B}' \right| \mathrm{d}m \geqslant \|1/W\|^{-2}_{L^\infty(\mathbb{T})} \cdot N_{\mathbb{D}}(\mathbf{B}); \tag{9}$$

we have also used the facts that $|F| = |W|$ on $\mathbb{T}$ (because $|\mathbf{B}| = |\mathcal{B}| = 1$ there) and $\|\mathbf{B}'\|_{L^1(\mathbb{T})} = N_{\mathbb{D}}(\mathbf{B})$. Similarly, we infer that

$$\mathrm{LHS} = \int_{\mathbb{D}} \left| W' \right|^2 \mathrm{d}A + n \int_{\mathbb{T}} |W|^2 \left| \mathcal{B}' \right| \mathrm{d}m \leqslant \left\| W' \right\|^2_{L^2(\mathbb{D})} + n \|W\|^2_{L^\infty(\mathbb{T})} \cdot N_{\mathbb{D}}(\mathcal{B}). \tag{10}$$

Finally, we combine the resulting inequalities from (9) and (10) to arrive at (6).

The strategy for proving (5) is largely similar to the above. This time, however, the Dirichlet space gets replaced by the Hardy–Sobolev space $H^1_1 := \{f\colon\ f' \in H^1\}$, and a result from [9] is employed concerning the canonical factorization in H^1_1.

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