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Partial Differential Equations/Mathematical Physics

On two-particle Anderson localization at low energies

Localisation d'Anderson pour un système à deux particules, à basses énergies

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ABSTRACT

We prove exponential spectral localization in a two-particle lattice Anderson model, with a short-range interaction and an external i.i.d. random potential, at sufficiently low energies. The proof is based on the multi-particle multi-scale analysis developed earlier in Chulaevsky and Suhov (2009) [4] in the case of high disorder. Our method applies to a larger class of random potentials than in Aizenman and Warzel (2009) [2] where dynamical localization was proved with the help of the fractional moment method.

RÉSUMÉ

On démontre la localisation spectrale exponentielle pour un modèle d'Anderson discret, avec interaction à courte portée dans un champ de potentiel aléatoire i.i.d., à basses énergies. La démonstration utilise l'analyse multi-échelle multi-particule développée dans Chulaevsky et Suhov (2009) [4] dans le cas de grand désordre. Cette méthode s'applique à une classe de potentiels aléatoires plus large que dans Aizenman et Warzel (2009) [2], où la localisation dynamique a été démontrée par la méthode des moments fractionnaires.

Version française abrégée

On étudie un système de deux particules quantiques avec interaction dans un milieu désordonné. Ce système est décrit par un Hamiltonien $H_{V,U}(\omega)$ agissant dans l'espace de Hilbert $\mathcal{H} := \ell^2(\mathbb{Z}^{2d})$, et de la forme suivante

$$H_{V,U} = \Delta + \sum_{j=1}^{2} V(x_j, \omega) + \mathbf{U},$$

où Δ est le Laplacien discret relatif au réseau $\mathbb{Z}^d \times \mathbb{Z}^d \cong \mathbb{Z}^{2d}$, i.e.,

$$\Delta\Psi(\mathbf{x}) = \sum_{\mathbf{y}\in\mathbb{Z}^{2d}:\ |\mathbf{y}|=1} \Psi(\mathbf{x}+\mathbf{y}), \quad \mathbf{x} = (x_1, x_2) \in \mathbb{Z}^{2d}$$

avec $|\mathbf{y}| := \|\mathbf{y}\|_\infty$. De plus, $V : \mathbb{Z}^d \times \Omega \to \mathbb{R}$ est un champ aléatoire i.i.d. sur $\mathbb{Z}^d$ relatif à un espace de probabilité $(\Omega, \mathfrak{F}, \mathbb{P})$, et $\mathbf{U}$ est l'opérateur de multiplication par une fonction bornée $\mathbf{U}(\mathbf{x}) = \mathbf{U}(x_1, x_2)$, non nécessairement symétrique.

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Le résultat principal de cette Note est le théorème suivant :

Théorème 0.1. *On suppose que V est un champ aléatoire réel, à valeurs indépendantes identiquement distribuées, et vérifiant la condition* (1)*. On suppose que le potentiel d'interaction* $\mathbf{U}$ *est borné et vérifie la condition* (2)*. On note* $E^0 = \inf\sigma(\mathbf{H})$.

Il existe alors un nombre réel $E^* > E^0$ *tel que le spectre de l'opérateur* $\mathbf{H}(\omega)$ *dans* $(-\infty, E^*]$ *soit purement ponctuel, et que toutes ses fonctions propres* $\Psi_n(\omega)$ *relatives aux valeurs propres* $E_n(\omega) \leqslant E^*$ *soient à décroissance exponentielle à l'infini* :

$$\left|\Psi_n(\mathbf{x})\right| \leqslant C_n(\omega)\mathrm{e}^{-m|\mathbf{x}|},$$

pour un $m > 0$ *non aléatoire.*

1. Introduction and main result

Consider the lattice $(\mathbb{Z}^d)^2 \cong \mathbb{Z}^{2d}$, $d \geqslant 1$. We will use the notations $\mathbb{D} = \{\mathbf{x} \in \mathbb{Z}^{2d}\colon\ \mathbf{x} = (x, x)\}$, $[\![a, b]\!] := [a, b] \cap \mathbb{Z}$. Vectors $\mathbf{x} = (x_1, x_2) \in \mathbb{Z}^d \times \mathbb{Z}^d$ will be identified with configurations of two distinguishable quantum particles in $\mathbb{Z}^d$. We denote by $|\cdot|$ the max-norm $\|\cdot\|_\infty$.

We study a system of two interacting lattice quantum particles in a disordered environment, described by a Hamiltonian $H_{V,U}(\omega)$ in the Hilbert space $\mathcal{H} := \ell^2(\mathbb{Z}^{2d})$ of the form

$$H_{V,U} = \Delta + \sum_{j=1}^{2} V(x_j, \omega) + \mathbf{U},$$

where Δ is the nearest-neighbor lattice Laplacian on $(\mathbb{Z}^d)^2 \cong \mathbb{Z}^{2d}$, i.e.

$$\Delta\Psi(\mathbf{x}) = \sum_{\mathbf{y}\in\mathbb{Z}^{2d}\colon\ |\mathbf{y}|=1} \Psi(\mathbf{x}+\mathbf{y}), \quad \mathbf{x} = (x_1, x_2) \in \mathbb{Z}^{2d},$$

$V : \mathbb{Z}^d \times \Omega \to \mathbb{R}$ is an i.i.d. random field on $\mathbb{Z}^d$ relative to some probability space $(\Omega, \mathfrak{F}, \mathbb{P})$, and $\mathbf{U}$ is the multiplication operator by a function $\mathbf{U}(\mathbf{x}) = \mathbf{U}(x_1, x_2)$ which we assume bounded — but not necessarily symmetric.

Aizenman and Warzel [2] proved by the fractional moment method — introduced in [1] for single-particle systems — spectral and dynamical localization for such Hamiltonians under the assumption that the marginal probability distribution of the random field V with i.i.d. (independent and identically distributed) values admits a bounded probability density ρ_V, satisfying some additional conditions.

In this note, using the Multi-Scale Analysis (MSA), we follow [4] and prove exponential localization under a much weaker assumption of log-Hölder continuity of the marginal distribution function F_V of the field V. Specifically, we require that for some $\beta \in (0, 1)$, some large enough $q > 0$, and all sufficiently large $L > 0$,

$$\sup_{a\in\mathbb{R}} \mathbb{P}\left\{V(0, \omega) \in \left[a, a + \mathrm{e}^{-L^\beta}\right]\right\} \leqslant L^{-q}. \tag{1}$$

The interaction potential is assumed to be bounded and to satisfy the following condition:

$$\text{there exists } r_0 \in [0, +\infty) \quad \text{such that} \quad |x_1 - x_2| > r_0 \quad \Rightarrow \quad \mathbf{U}(x_1, x_2) = 0. \tag{2}$$

We denote by $\sigma(\mathbf{H}(\omega))$ the spectrum of $\mathbf{H}(\omega)$. It follows from well-known results that the quantity

$$E^0 := \inf\sigma\left(\mathbf{H}(\omega)\right)$$

is non-random, although it may be infinite, e.g., for Gaussian random potentials.

Given an arbitrary finite lattice cube $\mathbf{C}_L(\mathbf{u}) := \{\mathbf{x} \in \mathbb{Z}^{2d} \mid |\mathbf{x} - \mathbf{u}| \leqslant L\}$, we will consider a finite-volume approximation of the Hamiltonian $\mathbf{H}$

$$\mathbf{H}_{\mathbf{C}_L(\mathbf{u})} = \mathbf{H}\restriction_{\ell^2(\mathbf{C}_L(\mathbf{u}))} \quad \text{with Dirichlet boundary conditions on } \partial\mathbf{C}_L(\mathbf{u}),$$

where the boundary is

$$\partial\mathbf{C}_L(\mathbf{u}) = \left\{\mathbf{v} \in \mathbb{Z}^{2d} \mid \operatorname{dist}\left(\mathbf{v}, \mathbb{Z}^{2d} \setminus \mathbf{C}_L(\mathbf{u})\right) = 1\right\}.$$

The main result of this note is the following:

Theorem 1.1. *Suppose V is a real i.i.d. random field satisfying condition* (1)*. Suppose also the interaction potential* $\mathbf{U}$ *is bounded and satisfies* (2)*. Let* $E^0 = \inf\sigma(\mathbf{H})$.

Then there exists $E^* > E^0$ *such that the spectrum of* $\mathbf{H}(\omega)$ *in* $(-\infty, E^*]$ *is pure point, and all its eigenfunctions* $\Psi_n(\omega)$ *with eigenvalues* $E_n(\omega) \leqslant E^*$ *are exponentially decaying at infinity*:

$$\left|\Psi_n(\mathbf{x})\right| \leqslant C_n(\omega)\mathrm{e}^{-m|\mathbf{x}|},$$

where $m > 0$ *is non-random.*

2. Proof scheme

Following [4], we use an adaptation to the two-particle interacting systems of the multi-scale analysis (MSA) which was earlier developed for single-particle models [8]. Given a finite cube $\mathbf{C}_L(\mathbf{u}) \subset \mathbb{Z}^{2d}$, introduce the resolvent of the operator $H_{\mathbf{C}_L(\mathbf{u})}$,

$$\mathbf{G}_{\mathbf{C}_L(\mathbf{u})}(E) := (H_{\mathbf{C}_L(\mathbf{u})} - E)^{-1}, \quad E \in \mathbb{R}.$$

Its matrix elements $\mathbf{G}_{\mathbf{C}_L(\mathbf{u})}(\mathbf{x}, \mathbf{y}; E)$ in the canonical basis $\delta_{\mathbf{x}}$ in $\ell^2(\mathbb{Z}^{2d})$ is usually called the (discrete) Green function of the operator $H_{\mathbf{C}_L(\mathbf{u})}$.

According to the general MSA approach, the exponential localization will be derived from Theorem 2.2 below. To formulate it, we need to introduce the following notion:

Definition 2.1. Let $m > 0$ and $E \in \mathbb{R}$. A cube $\mathbf{C}_L(\mathbf{u})$ is called (E, m)-*non-singular* ((E, m)-NS, in short) if

$$\max_{\mathbf{v} \in \partial \mathbf{C}_L(\mathbf{u})} \left| \mathbf{G}_{\mathbf{C}_L(\mathbf{u})}(\mathbf{u}, \mathbf{v}; E) \right| \leqslant \mathrm{e}^{-mL}.$$

Otherwise, it is called (E, m)-*singular* ((E, m)-S, in short).

Introduce the symmetry $S: (x_1, x_2) \mapsto (x_2, x_1)$ in the lattice $\mathbb{Z}^{2d}$ (here $x_1, x_2 \in \mathbb{Z}^d$) and define the "symmetrized" distance

$$d_S(\mathbf{x}, \mathbf{y}) = \min\{|\mathbf{x} - \mathbf{y}|, |S(\mathbf{x}) - \mathbf{y}|\}.$$

We will say that two lattice subsets $\mathbf{A}, \mathbf{B}$ are ℓ-*distant* if $d_S(\mathbf{A}, \mathbf{B}) > \ell$.

Further, given an integer $L_0 > 2$, define the sequence of integers $L_{k+1} = \lfloor L_k^{3/2} \rfloor$, $k \geqslant 0$. In the course of the MSA, it is required that L_0 be large enough.

Theorem 2.2. *Let $m > 0$. For any $p > 0$ there exists $E^* = E^*(p) > E^0$ such that for all $k \geqslant 0$ and any pair of $8L_k$-distant cubes $\mathbf{C}_{L_k}(\mathbf{u})$, $\mathbf{C}_{L_k}(\mathbf{v})$ the following bound holds true*:

$$\mathbb{P}\{\text{there exists } E \in [E^0, E^*] \text{ such that } \mathbf{C}_{L_k}(\mathbf{u}), \mathbf{C}_{L_k}(\mathbf{v}) \text{ are } (E, m)\text{-singular}\} \leqslant L_k^{-2p}, \tag{3}$$

provided L_0 is large enough.

The proof is based on induction in k. Note that the initial scale bound (for L_0 sufficiently large) uses the Combes–Thomas estimate and the "Lifshitz tails" phenomenon, essentially in the same way as for single-particle models [7], for the multi-particle structure of the potential energy is not relevant for such a bound. The inductive step is performed almost in the same way as in the case of high disorder (cf. [4]). It uses Wegner-type estimates proved in [3] and [6] (see [9] for the original Wegner estimate). Note, however, that unlike the high disorder regime, the value of the "mass" $m > 0$ may be small, depending upon the amplitude of the random potential V. Namely, if the random external potential has the form $gV(x; \omega)$, then the value of the "mass" $m = m(g) \to 0$ as $|g| \to 0$.

Unlike the single-particle case, the proof of (3) depends upon the geometry of the pair $\mathbf{C}_{L_k}(\mathbf{u})$, $\mathbf{C}_{L_k}(\mathbf{v})$. Namely, introduce the subset $\mathbb{D}_{r_0} := \{\mathbf{x} = (x_1; x_2) \in \mathbb{Z}^{2d}\colon\ |x_1 - x_2| \leqslant r_0\}$, and the following:

Definition 2.3. A 2-particle cube $\mathbf{C}_L(\mathbf{u})$ is called *diagonal* when $\mathbf{C}_L(\mathbf{u}) \cap \mathbb{D}_{r_0} \neq \emptyset$. Otherwise, it is called *non-diagonal.*

Property (3) is established separately for the following three types of pairs $\mathbf{C}_{L_{k+1}}(\mathbf{u})$, $\mathbf{C}_{L_{k+1}}(\mathbf{v})$ of separable cubes:

(i) Both are diagonal.
(ii) Both are non-diagonal.
(iii) One is diagonal, while the other is non-diagonal.

The next statement is a reformulation of [4, Theorem 1.2]; see the proof given in [4]. This theorem was earlier formulated in [8, Theorem 2.3] and [5, Section 1] for single-particle models.

Theorem 2.4. *Suppose that the bound* (3) *holds true for p large enough and some $E^* > E_0$.*

Then the spectrum of $\mathbf{H}(\omega)$ in $(-\infty, E^*]$ is pure point, and there exists a non-random number $m > 0$ such that all eigenfunctions $\Psi_n(\omega)$ of $\mathbf{H}(\omega)$ with eigenvalues $E_n(\omega) \leqslant E^*$ decay exponentially fast at infinity with rate m:

$$\left|\Psi_n(\mathbf{x})\right| \leqslant C_n(\omega)\mathrm{e}^{-m|\mathbf{x}|}.$$

Theorem 2.4 combined with Theorem 2.2 implies the main Theorem 1.1.

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