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Numerical Analysis

A remark on the optimality of adaptive finite element methods

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ABSTRACT

We show that the standard assumption on the smallness of the marking parameter θ in adaptive finite element methods can be avoided for the proof of the optimality of the algorithm. To this end we propose a new technique based on comparison of the solutions of different finite element spaces obtained by different refinements of a given mesh. We consider conforming and nonconforming low-order finite elements on triangular and tetrahedral meshes.

RÉSUMÉ

Nous démontrons que l'hypothèse de la petitesse du paramètre de marquage θ dans les méthodes d'éléments finis adaptatives peut être évitée dans la démonstration de l'optimalité de l'algorithme. Pour cela, nous introduisons une nouvelle technique basée sur la comparaison de différentes solutions correspondant à des espaces obtenus par différents raffinements d'un maillage donné. On considère des méthodes conformes et non conformes de bas degé sur des maillages en triangles et tetraèdres.

1. Introduction

Let $\Omega \subset \mathbb{R}^d$, $d = 2, 3$ be a polygonal domain and $u \in H_0^1(\Omega)$ the solution of the Poisson equation $-\Delta u = f$ in Ω.

Quasi-optimality of adaptive finite element methods means that the number of mesh-cells of the sequence of meshes constructed by the algorithm can be bounded by a power of the achieved accuracy in – up to a constant factor – best way. To be more precise, let $\mathcal{H}$ be a family of admissible meshes obtained by local mesh refinement and let $s > 0$ be the best possible speed of convergence for approximation of u, that is, given $N \in \mathbb{N}$, we can find a mesh $h \in \mathcal{H}$ with at most N cells such that best approximation u_h in the corresponding finite element space V_h yields

$$e_h := \left\|\nabla(u - u_h)\right\| \leqslant CN^{-s} \tag{1}$$

with an mesh-independent constant C, i.e. independent of $\mathcal{H}$. In order to obtain (1), certain regularity assumptions on u need to be valid (see below) and s depends on the dimension; for example classical a priori error analysis shows that if, $u \in H^2(\Omega)$ and V_h is the space of Courant elements on triangular meshes h, we have $s = 1/d$. The adaptive algorithm selects a sequence of meshes $\{h_k\}_{k=1,2,\dots}$ and corresponding finite element solutions $\{u_{h_k}\}_{k=1,2,\dots}$ with errors $\{e_k\}_{k=1,2,\dots}$, $e_k = \|\nabla(u - u_{h_k})\|$. Suppose that we have a convergence proof, $\lim_{k\to\infty} e_k = 0$, the quasi-optimality of the algorithm states that there exists a mesh-independent constant C such that $e_k \leqslant CN_k^{-s}$, where N_k denotes the number of cells of h_k.

Quasi-optimality of conforming methods has been proven in [4,9,6,2]; see also [1] for mixed and [3] for nonconforming finite elements. In the cited works, the marking is done by a bulk criterion of the form

$$\eta_h^2(\mathcal{M}) \geqslant \theta \eta_h^2, \tag{2}$$

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where h is a mesh in a family of admissible meshes $\mathcal{H}$ with cells $\mathcal{K}_h$ and $\mathcal{M} \subset \mathcal{K}_h$ denotes the set of cells marked for refinement; η_h is cell-wise a posteriori error estimator.

The main assumption in the standard proofs of quasi-optimality is that θ is small enough to satisfy a smallness condition of the type

$$\theta \leqslant C_1^{-1} C_2^{-1}, \tag{3}$$

where C_1 and C_2 are the constants in the lower and upper bounds of the estimator with respect to the error. It is obvious that $C_1 C_2 \geqslant 1$, and even in the case of error estimators giving sharp upper bounds as the hyper-cycling method [5], there is no hope for this product to be close to one, since the lower bound cannot be close to one [11]. Therefore (3) is a real restriction for applications.

In this note, we show that the condition $\theta < 1$ is sufficient for quasi-optimality. After introducing a precise definition of the adaptive algorithm in Section 2, this is first done for the Crouzeix–Raviart element in Section 3. The result is extended to the Courant element in Section 4.

2. Adaptive algorithm

We define the family of admissible meshes $\mathcal{H} = \mathcal{H}(h_0)$ in the following recursive way. Starting from an initial mesh h_0, we denote by $\mathcal{R}ef(h, \mathcal{M})$ the mesh resulting from a local refinement algorithm refining at least all cells $K \in \mathcal{M}$ for a given subset of marked cells $\mathcal{M} \subset \mathcal{K}_h$. If $h' = \mathcal{R}ef(h, \mathcal{M})$, we say that it is a local refinement of h. The refinement of additional cells is in general necessary in order to achieve certain regularity requirements. In this article, we make use of the following hypothesis on the local mesh refinement algorithm:

Hypothesis 2.1. *Let $h_k, k = 0, \ldots, n$ and $\mathcal{M}_k \subset \mathcal{K}_{h_k}, k = 0, \ldots, n-1$ be a sequence of locally refined meshes and the sets of marked cells, respectively, such that $h_{k+1} = \mathcal{R}ef(h_k, \mathcal{M}_k), k = 0, \ldots, n-1$. Then $\{h_k\}_k$ is uniformly shape regular and there exists a mesh-independent constant C_0 such that*

$$N_{h_n} \leqslant N_{h_0} + C_0 \sum_{k=0}^{n-1} \#\mathcal{M}_k. \tag{4}$$

Hypothesis 2.1 is known to be valid for the new-vertex-bisection algorithm in two [4] and three space dimensions [10].

Next we define the adaptive algorithm. In order to treat the case of data approximations, we employ the adaptive marking strategy presented in [2], which choses either the estimator or the data approximation for refinement, depending on their relative size. In the following algorithm we suppose to have an error estimator η_h and data approximation term μ_h, where the latter is independent of the discrete solution.

Adaptive finite element algorithm

(i) (Initialization) Choose an initial mesh h_0, $\gamma > 0$ marking parameters $0 < \theta, \sigma < 1$, and set $k = 0$.
(ii) (Solve) Solve the discrete problems on mesh h_n.
(iii) (Estimate) Compute the terms of the error estimator $(\eta_{h_k}(K))_K$ and data approximation $(\mu_{h_k}(K))_K$.
(iv) (Mark) Define a set $\mathcal{M}_k \subset \mathcal{K}_{h_k}$ of marked cells:
  – If $\mu_{h_k}^2 \leqslant \gamma\, \eta_{h_k}^2$ then define $\mathcal{M}_k := \mathcal{M}ark(\eta_{h_k}, \theta)$.
  – Else define $\mathcal{M}_k := \mathcal{M}ark(\mu_{h_k}, \sigma)$.
(v) (Refine) $h_{k+1} := \mathcal{R}ef(h_k, \mathcal{M}_k)$, set $k := k+1$ and go to step (Solve).

As in all cited works on adaptivity, the marking algorithm $\mathcal{M}ark(\xi, t)$ is based on the bulk criterion (2): Find the set of minimal cardinality such that $\xi(\mathcal{M})^2 \geqslant t\, \xi^2$ (where either $\xi = \eta_h$ or $\xi = \mu_h$).

3. Crouzeix–Raviart element

We consider the Crouzeix–Raviart space V_h of piecewise-liner functions, see [7]. Since $u_h \in V_h$ is not continuous in general, we use the piecewise gradient operator ∇_h defined by $\nabla_h u_h|_K := \nabla u_h|_K$ for all $K \in \mathcal{K}_h$. We consider the nonconforming finite element solution $u_h \in V_h$ defined by $\langle \nabla_h u_h, \nabla_h v_h \rangle = \langle f, v_h \rangle \ \forall v_h \in V_h$.

Quasi-optimality of the adaptive algorithm with bulk marking has been shown in [3] under the smallness assumption (3). In order to deal with the different finite element spaces required below, we start to give some notation. For given $h \in \mathcal{H}$ we denote the space of bounded and piecewise continuous functions by $C_h := \{v \in L^\infty(\Omega)\colon\ v|_K \in C(\overline{K})\ \forall K \in \mathcal{K}_h\}$. The set of sides (edges in 2D and faces in 3D) is denoted by $\mathcal{S}_h$. For a given interior side $S \in \mathcal{S}_h$, let n_S be a chosen unit normal vector. Let $v_h \in C_h$. For $x \in S$ we define $v_h^{in}(x) := \lim_{\varepsilon \searrow 0} v_h(x - \varepsilon n_S)$, $v_h^{ex}(x) := \lim_{\varepsilon \searrow 0} v_h(x + \varepsilon n_S)$, $\{v_h\}_S := \frac{1}{2}(v_h^{in}(x) + v_h^{ex}(x))$ and $[v_h](x) := v_h^{in}(x) - v_h^{ex}(x)$. For a boundary side, we set $n_S = n_{\partial\Omega}$ and $[v_h]_S(x) = v_h(x)$. The subscript S will be suppressed if this does not cause confusion.

Let h' be a refinement of h. Then we have $C_h \subset C_{h'}$ but $V_h \not\subset V_{h'}$. We extend the natural interpolation operator I_h in the following way: $I_h : C_h \to V_h$ $\int_S I_h v \, ds = \int_S \{v\} \, ds$ $\forall S \in \mathcal{S}_h$. For a refinement h' of h, integration by parts gives: $\langle \nabla_{h'}(u - I_{h'}u), \nabla_h v_h \rangle = 0$ $\forall v_h \in V_h$.

Let $S \in \mathcal{S}_h$. The estimator and data approximation for the nonconforming discretization are defined by

$$\eta_h(S) := |S|^{-1/2} \big\| [u_h] \big\|_S, \qquad \eta_h^2(\mathcal{M}) := \sum_{K \in \mathcal{M}} \sum_{S \subset \partial K \setminus \partial \Omega} \eta_h^2(S), \quad \eta_h := \eta_h(\mathcal{K}_h) \tag{5}$$

and

$$\mu_h(K) := |K|^{1/2} \|f\|_K, \qquad \mu_h^2(\mathcal{M}) := \sum_{K \in \mathcal{M}} \mu_h^2(K), \quad \mu_h := \mu_h(\mathcal{K}_h). \tag{6}$$

They satisfy the following global bounds (see for example [3] and references therein):

$$\big\| \nabla_h(u - u_h) \big\|^2 \leqslant C_1 \big(\eta_h^2 + \mu_h^2 \big), \quad \eta_h^2 \leqslant C_2 \big\| \nabla_h(u - u_h) \big\|^2. \tag{7}$$

The following local versions hold for a refined mesh h' of h with refined cells $\mathcal{R} \subset \mathcal{K}_h$:

$$\big\| \nabla_{h'}(u_{h'} - u_h) \big\|^2 \leqslant C_1 \big(\eta_h^2(\mathcal{R}) + \mu_h^2(\mathcal{R}) \big), \quad \eta_h^2(\mathcal{R}) \leqslant C_2 \big\| \nabla_{h'}(u_{h'} - u_h) \big\|^2. \tag{8}$$

In addition, the following decrease estimate for the data approximation term involving a mesh-independent constant $\kappa > 0$ holds:

$$\mu_{h'}^2 \leqslant \mu_h^2 - \kappa \mu_h^2(\mathcal{R}). \tag{9}$$

The geometrical convergence of the adaptive algorithm has been proven in [3] with respect to the error

$$e_h := \big\| \nabla_h(u - u_h) \big\|^2 + \beta \mu_h^2(f), \tag{10}$$

where $\beta > 0$ is sufficiently large. We then have existence of $\rho < 1$ such that

$$e_{h_{k+1}} \leqslant \rho \, e_{h_k}. \tag{11}$$

Our main tool for the improvement of the optimality result is the following inequality:

Lemma 3.1. *Let $h \in \mathcal{H}$ and $\mathcal{M}_i \subset \mathcal{K}_h$, $i = 1, 2$, such that the set of refined cells $\mathcal{R}_i$ satisfy $\mathcal{R}_1 \cap \mathcal{R}_2 = \emptyset$. Then it holds with $h_i := \mathcal{R}ef(h, \mathcal{M}_i)$ and the corresponding finite element solutions u_{h_i}, $i = 1, 2$, that*

$$\big\| \nabla_{h_1}(u_{h_1} - u_h) \big\| \leqslant \big\| \nabla_{h_2}(u - u_{h_2}) \big\| + C \mu_h. \tag{12}$$

We do not give the proof here, but remark that for approximation with stronger orthogonality than finite elements, such as wavelets and Fourier series, one has

$$\big\| \nabla_{h_1}(u_{h_1} - u_h) \big\| = \big\| \nabla_{h_2}(u - u_{h_2}) \big\|. \tag{13}$$

In order to express the optimal complexity of the algorithm, we introduce some notation from nonlinear approximation theory, see [4,8]. Let $\mathcal{H}_N$ be the set of all meshes h which satisfy $N_h \leqslant N$.

Next we define the approximation class $\mathcal{W}^s := \{(u, f) \in H_0^1(\Omega) \times L^2(\Omega){:}\ \|(u, f)\|_{\mathcal{W}^s} < +\infty\}$ with $\|(u, f)\|_{\mathcal{W}^s}^2 := \sup_{N \geqslant N_0} N^s \inf_{h \in \mathcal{H}_N} (\|\nabla_h(u - u_h)\|^2 + \mu_h^2(f))$. We say that an adaptive finite element method realizes the optimal convergence rate if, whenever $(u, f) \in \mathcal{W}^s$, there exists an absolute constant C such that the generated sequence of triangulations $\{h_k\}$ with dimensions N_k and corresponding approximations u_{h_k} satisfies

$$\big\| \nabla_{h_k}(u - u_{h_k}) \big\|^2 + \mu_k^2(f) \leqslant C N_k^{-s}. \tag{14}$$

Alternatively, setting $\varepsilon_{h_k} := \|\nabla_{h_k}(u - u_{h_k})\|^2 + \mu_k^2(f)$, we may ask for the complexity estimate

$$N_k \leqslant C \varepsilon_{h_k}^{-1/s}. \tag{15}$$

Theorem 3.2. *The adaptive algorithm realizes the optimal convergence rate, if $\theta < 1$ and if γ is chosen small enough.*

The idea of the proof is the following. Given iteration k, there exists a mesh $h^* \in \mathcal{H}$ such that with $\lambda > 0$ to be chosen below there holds

$$\varepsilon_{h^*} \leqslant \lambda \varepsilon_{h_k} \quad \text{and} \quad N_{h^*} \leqslant C \varepsilon_{h_k}^{-1/s}. \tag{16}$$

Following [9], we can assume that h^* is a refinement of h_k. Suppose for simplicity that $\mu_h = 0$. The essential part of the proof is to show that

$$\eta_{h_k}^2 \big(\mathcal{M}^* \big) \geqslant \theta \eta_{h_k}^2. \tag{17}$$

Let now $\Delta \subset \mathcal{K}_h$ be the set of neighbors of $\mathcal{M}^*$ and set $\widehat{\mathcal{M}} := \mathcal{K}_{h_k} \setminus (\mathcal{M}^* \cup \Delta)$. We wish to show that

$$\eta^2_{h_k}(\widehat{\mathcal{M}}) \leqslant (1-\theta)\eta^2_{h_k}. \tag{18}$$

To do so, let $\hat{h} = \mathcal{R}ef(h, \widehat{\mathcal{M}})$. We now employ Lemma 3.1 with $h_1 = \hat{h}$ and $h_2 = h^*$.

We have by the local upper bound, (12), (16), and the global lower bound

$$\eta^2_{h_k}(\widehat{\mathcal{M}}) \leqslant C_2 \left\| \nabla_{\hat{h}}(u_{\hat{h}} - u_h)\right\|^2 \leqslant 2C_2 \left\| \nabla_{h^*}(u - u_{h^*})\right\|^2 \leqslant 2C_2\lambda \left\| \nabla_{h_k}(u - u_{h_k})\right\|^2 \leqslant 2C_2C_1\lambda\eta^2_{h_k}.$$

Choosing λ sufficiently small implies (18) and therefore $\eta^2_{h_k}(\mathcal{M}^* \cup \Delta) \geqslant \theta\,\eta^2_{h_k}$. By the optimality of $\mathcal{M}_k$ and the fact that the numbers of neighbors of a set is bounded by the cardinality of the set, we finally get $\#\mathcal{M}_k \leqslant \#\mathcal{M}^* + \#\Delta \leqslant 4\#\mathcal{M}^* \leqslant C\varepsilon_k^{-1/s}$, which is the required estimate for the complexity proof (see [3] for details).

4. Courant elements

We now consider the Courant space W_h of continuous piecewise-liner functions. The solution of the discrete problem $u^c_h \in W_h$ is defined by $\langle \nabla u^c_h, \nabla v_h\rangle = \langle f, v_h\rangle\ \forall v_h \in W_h$. Quasi-optimality of the adaptive algorithm with bulk marking has been shown in [2] under the smallness assumption on θ (3). In order to get rid of this assumption, we use the following result.

Lemma 4.1. *Let $h \in \mathcal{H}$ and $\mathcal{M}_i \subset \mathcal{K}_h$, $i = 1, 2$, such that the set of refined cells $\mathcal{R}_i$ satisfy $\mathcal{R}_1 \cap \mathcal{R}_2 = \emptyset$. Then there exists a mesh-independent const C_3 such that with $h_i := \mathcal{R}ef(h, \mathcal{M}_i)$*

$$\left\| \nabla\left(u^c_{h_1} - u^c_h\right)\right\| \leqslant C_3 \left\| \nabla\left(u - u^c_{h_2}\right)\right\|. \tag{19}$$

The idea of the proof is to compare with the nonconforming finite element solutions.

Let π_K denote the $L^2(K)$-projection on the constants. The estimator and data approximation for the conforming discretization are defined by

$$\eta^c_h(K) := |K|\|f + \Delta u_h\|_K + |K|^{1/2}\left\| \left[\frac{\partial u^c_h}{\partial n}\right]\right\|_{\partial K\setminus\partial\Omega}, \quad \eta^c_h(\mathcal{M})^2 := \sum_{K\in\mathcal{M}} \eta^c_h(K)^2 \tag{20}$$

and

$$\mu^c_h(K) := |K|^{1/2}\|f - \pi_K f\|_K, \quad \mu^c_h(\mathcal{M})^2 := \sum_{K\in\mathcal{M}} \mu^c_h(K)^2. \tag{21}$$

As before we set for abbreviation $\eta^c_h := \eta^c_h(\mathcal{K}_h)$ and $\mu^c_h := \mu^c_h(\mathcal{K}_h)$. We have the following global bounds

$$\left\| \nabla\left(u - u^c_h\right)\right\|^2 \leqslant C_1\left(\eta^c_h\right)^2, \quad \eta^{c\,2}_h \leqslant C_2\left(\left\| \nabla\left(u - u^c_h\right)\right\|^2 + \left(\mu^c_h\right)^2\right). \tag{22}$$

The following local versions hold for a refined mesh h' of h with refined cells $\mathcal{R} \subset \mathcal{K}_h$:

$$\left\| \nabla\left(u^c_{h'} - u^c_h\right)\right\|^2 \leqslant C_1\left(\eta^c_h\right)^2(\mathcal{R}), \quad \eta^{c\,2}_h(\mathcal{R}) \leqslant C_2\left(\left\| \nabla\left(u^c_{h'} - u^c_h\right)\right\|^2 + \left(\mu^c_h\right)^2(\mathcal{R})\right). \tag{23}$$

The geometrical convergence of the adaptive algorithm has been proven in [2] with respect to the error

$$e_h := \left\| \nabla_h(u - u_h)\right\|^2 + \beta\mu^c_h(f)^2, \tag{24}$$

where $\beta > 0$ is sufficiently large. We then have existence of $\rho < 1$ such that $e_{h_{k+1}} \leqslant \rho e_{h_k}$.

The approximation class $\mathcal{W}^s$ is defined as in the nonconforming space, replacing the discrete spaces.

Theorem 4.2. *The adaptive algorithm realizes the optimal convergence rate, if $\theta < 1$ and $\gamma < (1-\theta)/(2C_2)$.*

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