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On a waveguide with an infinite number of small windows

Sur un guide d'onde avec un nombre infini de petites fentes

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A R T I C L E I N F O

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A B S T R A C T

We consider a waveguide modeled by the Laplacian in a straight planar strip with the Dirichlet condition on the upper boundary, while on the lower one we impose periodically alternating boundary conditions with a small period. We study the case when the homogenization leads us to the Neumann boundary condition on the lower boundary. We establish the uniform resolvent convergence and provide the estimates for the rate of convergence. We construct the two-terms asymptotics for the first band functions of the perturbed operator and also the complete two-parametric asymptotic expansion for the bottom of its spectrum.

R É S U M É

On considère un guide d'onde modélisé par le Laplacien dans une bande horizontale, avec des conditions de Dirichlet sur le bord supérieur et des conditions du type Dirichlet et Neumann qui alternent périodiquement sur le bord inférieur. La période est considérée petite et on étudie le problème de l'homogénéisation : on démontre la convergence en norme de la résolvante vers la résolvante du Laplacien avec des conditions de Neumann sur tout le bord inférieur et on obtient des estimations du taux de convergence. Ensuite on donne les deux premiers termes du développement asymptotique des valeurs propres de l'opérateur perturbé, ainsi que le développement asymptotique complet du bas de son spectre.

1. Introduction

In the present Note we consider a model of a quantum waveguide with an infinite number of windows. This model was suggested in [3]. It is the Laplacian in a straight strip subject to the Dirichlet condition except windows. The windows are located on the lower boundary and modeled by the Neumann condition on an infinite periodic set of small segments

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located closely each to other. In fact, this is the waveguide with a frequent alternation of the boundary conditions, which is a singular perturbation from the homogenization theory for bounded domains, see, for instance, [1,5,7–9] and the references therein. The cited papers can be considered as the motivation for the present note. One more motivation comes from the waveguide theory, where the waveguides with a finite number of windows were studied intensively, see, for instance, [2,4,6], and the references therein. At the same time, our model differs very much by the occurring phenomena. The main distinction to the problems in [1,5,7–9] is that we consider an unbounded domain producing a non-trivial essential spectrum. In contrast to [2,4,6], the essential spectrum is not stable, but sensitive to the alternation of the boundary conditions.

We consider the described model assuming that the homogenized operator has the Neumann condition on the lower boundary instead of the alternating ones. This is the main distinction to [3], where the homogenized condition was the Dirichlet one. Our main results are the uniform resolvent convergence and the estimates for the rate of convergence, and the two-terms asymptotic expansions for the first band functions and the complete two-parametric expansion for the bottom of the spectrum. We observe that for bounded domains [5,7] the uniform resolvent convergence was not known. Although our results are similar to that of [3], they are of a different nature due to another homogenized boundary condition.

The first important distinction to [3] is the estimates for the rate of convergence for the perturbed resolvent. In [3] a good estimate was obtained for the difference of the perturbed and homogenized resolvents considered operators from L_2 into W_2^1. In our case we obtain a similar good estimate only if we employ certain boundary corrector and consider the difference of the perturbed resolvent and the resolvent of a model operator still depending on a small parameter. Omitting the corrector or considering the homogenized resolvent, one worsens the rate of convergence. This situation is close to that in homogenization of the operators with fast oscillating coefficients in unbounded domains, see, for instance, [10,11].

The second important distinction to [3] is the asymptotics for the bottom of the spectrum consisting of the leading term with an *exponentially* small (both in ε and μ) error. The leading term is shown to be a *holomorphic* in ε and μ function. It means that the power in ε and μ part of the asymptotics for the bottom of the spectrum can be summed. This is a completely new result for the problems with frequent alternation of boundary condition, since before all the expansions were just asymptotic, cf. [1,9].

2. Formulation of the problem and the main results

Let $x = (x_1, x_2)$ be Cartesian coordinates in $\mathbb{R}^2$, $\Omega := \{x\colon\ 0 < x_2 < \pi\}$ be a straight strip of width π. By ε we denote a small positive parameter, and $\eta = \eta(\varepsilon)$ is a function with values in $(0, \pi/2)$. We indicate by Γ_+ and Γ_- the upper and lower boundary of Ω, and we partition Γ_- into two subsets, $\gamma_\varepsilon := \{x\colon\ |x_1 - \varepsilon\pi j| < \varepsilon\eta,\ x_2 = 0,\ j \in \mathbb{Z}\}$ and $\Gamma_\varepsilon := \Gamma_- \setminus \overline{\gamma_\varepsilon}$. By $\mathcal{H}_\varepsilon$ we indicate the Laplacian in $L_2(\Omega)$ subject to the Dirichlet boundary condition on $\Gamma_+ \cup \gamma_\varepsilon$ and to the Neumann one on Γ_ε. This operator is introduced as the self-adjoint one in $L_2(\Omega)$ associated with the sesquilinear form $(\nabla u, \nabla v)_{L_2(\Omega)}$ on $\mathring{W}_2^1(\Omega, \Gamma_+ \cup \gamma_\varepsilon)$, where $\mathring{W}_2^1(Q, S)$ indicates the subset of the functions in $W_2^1(Q)$ having zero trace on the curve S. The aim is to study the asymptotic behavior of the resolvent and the spectrum of $\mathcal{H}_\varepsilon$ as $\varepsilon \to +0$.

Let $\mathcal{H}^{(\mu)}$ be the self-adjoint operator in $L_2(\Omega)$ associated with the sesquilinear form $(\nabla u, \nabla v)_{L_2(\Omega)} + \mu(u, v)_{L_2(\partial\Omega)}$ on $\mathring{W}_2^1(\Omega, \Gamma_+)$, where $\mu \geqslant 0$ is a constant. The spectrum of $\mathcal{H}^{(0)}$ is purely essential and coincides with $[\frac{1}{4}, +\infty)$. By $\|\cdot\|_{X\to Y}$ we denote the norm of an operator acting from X to Y.

Our first theorem establishes the uniform resolvent convergence for $\mathcal{H}_\varepsilon$.

Theorem 2.1. *Suppose*

$$\varepsilon \ln \eta(\varepsilon) \to -\infty, \quad \mu = \mu(\varepsilon) := -\frac{1}{\varepsilon \ln \eta(\varepsilon)} \to +0 \quad \varepsilon \to +0. \tag{1}$$

Then there exists a boundary corrector $W = W(x, \varepsilon, \mu)$ which can be defined explicitly such that

$$\big\|(\mathcal{H}_\varepsilon - \mathrm{i})^{-1} - (1 + W)\big(\mathcal{H}^{(\mu)} - \mathrm{i}\big)^{-1}\big\|_{L_2(\Omega)\to W_2^1(\Omega)} \leqslant C\varepsilon\mu|\ln \varepsilon\mu|,$$

$$\big\|(\mathcal{H}_\varepsilon - \mathrm{i})^{-1} - \big(\mathcal{H}^{(\mu)} - \mathrm{i}\big)^{-1}\big\|_{L_2(\Omega)\to L_2(\Omega)} \leqslant C\varepsilon\mu|\ln \varepsilon\mu|,$$

$$\big\|(\mathcal{H}_\varepsilon - \mathrm{i})^{-1} - \big(\mathcal{H}^{(0)} - \mathrm{i}\big)^{-1}\big\|_{L_2(\Omega)\to W_2^1(\Omega)} \leqslant C\mu^{1/2}, \qquad \big\|(\mathcal{H}_\varepsilon - \mathrm{i})^{-1} - \big(\mathcal{H}^{(0)} - \mathrm{i}\big)^{-1}\big\|_{L_2(\Omega)\to L_2(\Omega)} \leqslant C\mu,$$

where the constants C are independent of ε and μ. The spectrum of $\mathcal{H}_\varepsilon$ converges to that of $\mathcal{H}^{(0)}$. Namely, if $\lambda \notin [\frac{1}{4}, +\infty)$, then $\lambda \notin \sigma(\mathcal{H}_\varepsilon)$ for ε small enough. If $\lambda \in [\frac{1}{4}, +\infty)$, then there exists $\lambda_\varepsilon \in \sigma(\mathcal{H}_\varepsilon)$ so that $\lambda_\varepsilon \to \lambda$ as $\varepsilon \to +0$.

Since the alternation of the boundary conditions for $\mathcal{H}_\varepsilon$ is periodic, this operator is periodic, too. We employ the Floquet–Bloch decomposition to study its spectrum. Denote $\Omega_\varepsilon := \{x\colon\ |x_1| < \varepsilon\pi/2,\ 0 < x_2 < \pi\}$, $\mathring{\gamma}_\varepsilon := \partial\Omega_\varepsilon \cap \gamma_\varepsilon$, $\mathring{\Gamma}_\varepsilon := \partial\Omega_\varepsilon \cap \Gamma_\varepsilon$, $\mathring{\Gamma}_\pm := \partial\Omega_\varepsilon \cap \Gamma_\pm$. By $\mathring{\mathcal{H}}_\varepsilon(\tau)$ we indicate the self-adjoint operator in $L_2(\Omega_\varepsilon)$ associated with the sesquilinear form $((\mathrm{i}\partial_{x_1} - \tau\varepsilon^{-1})u, (\mathrm{i}\partial_{x_1} - \tau\varepsilon^{-1})v)_{L_2(\Omega_\varepsilon)} + (\partial_{x_2}u, \partial_{x_2}v)_{L_2(\Omega_\varepsilon)}$ on $\mathring{W}_{2,per}^1(\Omega_\varepsilon, \mathring{\Gamma}_+ \cup \mathring{\gamma}_\varepsilon)$, $\tau \in [-1, 1)$. Here $\mathring{W}_{2,per}^1(\Omega_\varepsilon, \mathring{\Gamma}_+ \cup \mathring{\gamma}_\varepsilon)$ is the set of the functions in $\mathring{W}_2^1(\Omega_\varepsilon, \mathring{\Gamma}_+ \cup \mathring{\gamma}_\varepsilon)$ satisfying periodic boundary conditions on the lateral boundaries of Ω_ε. The spectrum of $\mathring{\mathcal{H}}_\varepsilon(\tau)$ consists of discrete eigenvalues only. We indicate them by $\lambda_n(\tau,\varepsilon)$, $n \in \mathbb{N}$ and arrange in the ascending order counting multiplicity. By [3, Lm. 4.1] we know that the spectrum of $\mathcal{H}_\varepsilon$ contains its essential part only,

$$\sigma(\mathcal{H}_\varepsilon) = \sigma_e(\mathcal{H}_\varepsilon) = \bigcup_{n=1}^{\infty} \{\lambda_n(\tau,\varepsilon):\ \tau \in [-1,1)\}.$$

Let $\mathfrak{L}_\varepsilon$ be the subspace of $L_2(\Omega_\varepsilon)$ consisting of the functions independent of x_1, and $\mathcal{Q}_\mu$ be the self-adjoint operator in $\mathfrak{L}_\varepsilon$ associated with the sesquilinear form $(u',v')_{L_2(0,\pi)} + \mu u(0)\overline{v(0)}$ on $\mathring{W}_2^1((0,\pi),\{\pi\})$. Our next theorem describes the uniform resolvent convergence for $\mathring{\mathcal{H}}_\varepsilon(\tau)$.

Theorem 2.2. *Let $|\tau| < 1-\varkappa$, where $0<\varkappa<1$ is a fixed constant and suppose (1). Then for sufficiently small ε the estimate*

$$\big\| \big(\mathring{\mathcal{H}}_\varepsilon(\tau) - \tau^2\varepsilon^{-2}\big)^{-1} - \mathcal{Q}_\mu^{-1} \oplus 0 \big\|_{L_2(\Omega_\varepsilon)\to L_2(\Omega_\varepsilon)} \leqslant C\varkappa^{-1/2}\big(\varepsilon^{1/2}\mu + \varepsilon\big) \tag{2}$$

holds true, where the constant C is independent of ε, μ, and $\varkappa$.

The next result gives the two-terms asymptotics for the first band functions $\lambda_n(\tau,\varepsilon)$.

Theorem 2.3. *Let the hypothesis of Theorem 2.2 holds true. Then given any N, for $\varepsilon < 2\varkappa^{1/2}N^{-1}$ the eigenvalues $\lambda_n(\tau,\varepsilon)$, $n = 1,\dots,N$, satisfy the relations*

$$\lambda_n(\tau,\varepsilon) = \tau^2\varepsilon^{-2} + \Lambda_n(\mu) + R_n(\tau,\varepsilon,\mu), \qquad \big|R_n(\tau,\varepsilon,\mu)\big| \leqslant C\varkappa^{-1/2}n^4\varepsilon^{1/2}\mu, \tag{3}$$

where $\Lambda_n(\mu)$, $n = 1,\dots,N$, are the first N eigenvalues of $\mathcal{Q}_\mu$, and the constant C is the same as in (2). The eigenvalues $\Lambda_n(\mu)$ solve the equation $\sqrt{\Lambda}\cos\sqrt{\Lambda}\pi + \mu\sin\sqrt{\Lambda}\pi = 0$, are holomorphic w.r.t. μ, and

$$\Lambda_n(\mu) = \left(n-\frac{1}{2}\right)^2 + \frac{\mu}{\pi(n-\frac{1}{2})} + \mathcal{O}\big(\mu^2\big), \quad \mu \to +0. \tag{4}$$

The asymptotics (3) imply that the length of the first zones of the spectrum of $\mathcal{H}_\varepsilon$ are of order $\mathcal{O}(\varepsilon^{-2})$ and they overlap. It implies that if they exist, all gaps in the spectrum of $\mathcal{H}_\varepsilon$ "run" to infinity as $\varepsilon \to +0$ with the speed at least $\mathcal{O}(\varepsilon^{-2})$.

Let ζ be the Riemann zeta-function, and $\theta(\beta)$ be a holomorphic function defined by

$$\theta(\beta) := -\sum_{j=1}^{+\infty} \frac{1}{n\sqrt{4j^2-\beta}(2j+\sqrt{4j^2-\beta})} = -\sum_{j=1}^{+\infty} \frac{(2j-1)!!\zeta(2j+1)}{8^j j!}\beta^{j-1}. \tag{5}$$

Our last theorem gives the complete asymptotic expansion for the bottom of the spectrum of $\mathcal{H}_\varepsilon$.

Theorem 2.4. *For ε small enough, the first eigenvalue $\lambda_1(\tau,\varepsilon)$ attains its minimum at $\tau = 0$, i.e., $\inf\limits_{\tau\in[-1,1)} \lambda_1(\tau,\varepsilon) = \lambda_1(0,\varepsilon)$. The asymptotics*

$$\lambda_1(0,\varepsilon) = \Lambda(\varepsilon,\mu) + \mathcal{O}\big(\mu\varepsilon^{-1/2}\mathrm{e}^{-2\varepsilon^{-1}} + \varepsilon^{1/2}\eta^{1/2}\big) \tag{6}$$

holds true, where $\Lambda(\varepsilon,\mu)$ is the real solution to the equation

$$\sqrt{\Lambda}\cos\sqrt{\Lambda}\pi + \mu\sin\sqrt{\Lambda\pi} - \varepsilon^3\mu\Lambda^{3/2}\theta\big(\varepsilon^2\Lambda\big)\cos\sqrt{\Lambda}\pi = 0, \qquad \Lambda(\varepsilon,\mu) = \Lambda_1(\mu) + o(1), \quad \varepsilon \to 0. \tag{7}$$

The function $\Lambda(\varepsilon,\mu)$ is jointly holomorphic w.r.t. ε and μ. It can be represented as the series

$$\Lambda(\varepsilon,\mu) = \Lambda_1(\mu) + \mu^2\sum_{j=1}^{+\infty}\varepsilon^{2j+1}K_{2j+1}(\mu) + \mu^3\sum_{j=3}^{+\infty}\varepsilon^{2j}K_{2j}(\mu), \tag{8}$$

where the functions $K_j(\mu)$ are holomorphic w.r.t. μ, and, in particular,

$$K_3(\mu) = -\frac{\zeta(3)}{4}\frac{\Lambda_1^2(\mu)}{\pi\Lambda_1(\mu)+\mu+\pi\mu^2}, \qquad K_5(\mu) = -\frac{3\zeta(5)}{64}\frac{\Lambda_1^3(\mu)}{\pi\Lambda_1(\mu)+\mu+\pi\mu^2},$$

$$K_6(\mu) = \frac{\zeta(3)^2}{64}\frac{\Lambda_1^3(\mu)(2\pi^2\Lambda_1^2(\mu) + 7\pi\mu\Lambda_1(\mu) + 2\pi^2\mu^2\Lambda_1(\mu) + 7\mu^2 + 7\pi\mu^3)}{(\pi\Lambda_1(\mu)+\mu+\pi\mu^2)^3}.$$

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