Mathematical Analysis

Multiplier sequences and logarithmic mesh

Multiplicateurs et mailles logarithmiques

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A R T I C L E   I N F O

A B S T R A C T

In this Note we prove a new result about (finite) multiplier sequences, i.e. linear operators acting diagonally in the standard monomial basis of $\mathbb{R}[x]$ and sending polynomials with all real roots to polynomials with all real roots. Namely, we show that any such operator does not decrease the logarithmic mesh when acting on an arbitrary polynomial having all roots real and of the same sign. The logarithmic mesh of such a polynomial is defined as the minimal quotient of its consecutive roots taken in the non-decreasing order of their absolute values.

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R É S U M É

Les multiplicateurs considérés dans cette Note sont les opérateurs linéaires qui agissent diagonalement sur $\mathbb{R}[x]$ muni de sa base standard (les monômes) et qui transforment les polynômes à racines réelles en polynômes à racines réelles. Nous montrons qu'un tel opérateur, appliqué à un polynôme dont toutes les racines sont réelles et de même signe, ne diminue pas la maille logarithmique, c'est-à-dire le minimum du quotient de deux racines consécutives dans l'ordre croissant des valeurs absolues.

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1. Introduction

Denote by $\mathcal{HP} \subset \mathbb{R}[x]$ the set of all real-rooted (also referred to as *hyperbolic*) polynomials. A linear operator $T : \mathbb{R}[x] \to \mathbb{R}[x]$ is called a *real rootedness preserver* or a *hyperbolicity preserver* if it preserves $\mathcal{HP}$. (We will also use the short-hand 'HPO' for such operators. A characterization of hyperbolicity preservers was recently obtained in [1].) Given a real-rooted polynomial $p(x) \in \mathcal{HP}$ denote by mesh(p) its mesh, i.e. the minimal distance between its real roots. (If $p(x)$ has a double real root then mesh$(p) = 0$.)

One of the (very few known) results about linear operators not decreasing mesh is presented below and is due originally to M. Riesz but was written down by A. Stoyanoff, see [6].

Proposition 1. *For any hyperbolic polynomial p and any real λ one has*

$$\text{mesh}(p - \lambda p') \geqslant \text{mesh}(p).$$

Recall that the well-known Hermite–Poulain theorem (see [4], p. 4) claims that a finite order linear differential operator $T = a_0 + a_1 \frac{d}{dx} + \cdots + a_k \frac{d^k}{dx^k}$ with constant coefficients is a hyperbolicity preserver iff its symbol polynomial $Q(t) = a_0 + a_1 t + \cdots + a_k t^k$ is hyperbolic.

Thus, Proposition 1 combined with the Hermite–Poulain theorem immediately imply the following statement:

Theorem 2. *Any hyperbolicity preserving differential operator with constant coefficients does not decrease the mesh of hyperbolic polynomials.*

For the sake of completeness and due to the fact that [6] is hardly available nowadays we reprove Theorem 2 below. Our main interest in this Note is to find an analog of Proposition 1 and Theorem 2 for another famous class of HPO, namely, for the so-called multiplier sequences characterized by G. Pólya and J. Schur in [5]. The relevant basic notions are as follows.

Given a sequence $\mathcal{A} = \{\alpha_n\}_{n=0}^\infty$ of real or complex numbers we denote by $T_\mathcal{A}$ the linear operator acting diagonally in the monomial basis of $\mathbb{C}[x]$ by $T_\mathcal{A}(x^i) = \alpha_i x^i$. We refer to $T_\mathcal{A}$ as the *diagonal operator corresponding to $\mathcal{A}$*.

We call a sequence $\mathcal{A} = \{\alpha_n\}_{n=0}^\infty$ of real numbers a *multiplier sequence of the 1st kind*, if its diagonal operator $T_\mathcal{A}$ preserves $\mathcal{HP}$, i.e. sends an arbitrary real-rooted polynomial to a real-rooted polynomial. The above sequence $\mathcal{A}$ is called a *multiplier sequence of the 2nd kind*, if the above $T_\mathcal{A}$ sends an arbitrary real-rooted polynomial whose roots are all of the same sign to a real-rooted polynomial.

The following fundamental criterion was found in [5]:

Theorem 3. *Let $\mathcal{A} = \{\alpha_n\}_{n=0}^\infty$ be a sequence of real numbers and let $T_\mathcal{A} : \mathbb{R}[x] \to \mathbb{R}[x]$ be the corresponding diagonal operator. Define $\Phi(t)$ to be the formal power series*

$$\Phi(t) = \sum_{n=0}^\infty \frac{\alpha_n}{n!} t^n.$$

The following assertions are equivalent:

(i) *$\mathcal{A}$ is a multiplier sequence of the first kind.*
(ii) *$\Phi(t)$ defines an entire function which is the limit, uniformly on compact sets, of polynomials with only real zeros of the same sign.*
(iii) *Either $\Phi(t)$ or $\Phi(-t)$ is an entire function that can be written as*

$$\Phi(t) = Ct^k e^{at} \prod_{n=1}^\infty (1 + \gamma_n t),$$

where $k \in \mathbb{N}$, $C \in \mathbb{R}$, a; $\gamma_n \geqslant 0$ for all $n \in \mathbb{N}$ and $\sum_{n=1}^\infty \gamma_n < \infty$.
(iv) *For all non-negative integers k the polynomial $T[(1+x)^k]$ is hyperbolic with all zeros of the same sign.*

Let us also recall the following analog of the Pólya–Schur theorem in finite degrees.

Consider a diagonal (in the monomial basis) operator $T : \mathbb{R}_k[x] \to \mathbb{R}_k[x]$ acting by multiplication of x^j by γ_j, $j = 0, \ldots, k$. A diagonal hyperbolicity preserver T will be referred to as a *multiplier sequence of length $k+1$* or simply a *finite multiplier sequence*. Denoted by $M_k \subset \mathbb{R}^{k+1}$ the set of all finite multiplier sequences of length $k+1$.

The following result was originally proved in [3], Theorem 3.7, see also [2], Theorem 3.1:

Theorem 4. *For $T \in \mathbb{GL}(\mathbb{R}_k[x])$ the following two conditions are equivalent*:

(i) *T is a multiplier sequence of length $k+1$.*
(ii) *The polynomial $Q_T(t) = \sum_{j=0}^k \binom{k}{j} \gamma_j t^j$ has all real zeros of the same sign.*

One can identify the semigroup of all finite multiplier sequences of length $k+1$ with the set of polynomials of degree k having all real roots of the same sign. The usual multiplication of diagonal matrices then corresponds to the so-called Schur–Szegő multiplication of polynomials, see e.g. [7]. Namely, the *Schur–Szegő product $P * Q$* of two polynomials $P(x) = \sum_{j=0}^k \binom{k}{j} a_j x^j$ and $Q(x) = \sum_{j=0}^k \binom{k}{j} b_j x^j$ equals

$$P * Q = \sum_{j=0}^k \binom{k}{j} a_j b_j x^j.$$

Given a polynomial P of degree k with all real roots of the same sign order these roots as $|x_1| \leqslant |x_2| < \cdots < |x_k|$ and define the *logarithmic mesh* of P as

$$\text{lmesh}(P) = \min_{j=1,\dots,k-1} \frac{|x_{j+1}|}{|x_j|}.$$

Obviously, for any polynomial with all real roots of the same sign one has that $\text{lmesh}(P) \geqslant 1$ and $\text{lmesh}(P) = 1$ if and only P has a multiple (real) root. We can now formulate the new results of this Note. An analog of Proposition 1 is as follows.

Proposition 5. *For any $\lambda > 0$ the differential operator $T(p(x)) = \lambda p + x p'$ has the property*

$$\text{lmesh}(T(p)) \geqslant \text{lmesh}(p),$$

where p is an arbitrary polynomial p with all real roots of the same sign.

The latter proposition can be generalized to the Schur–Szegő product of two polynomials of the same degree. Namely, the following statement holds.

Theorem 6. *Given two polynomials P and Q of degree k with all roots of the same sign one has*

$$\text{lmesh}(P * Q) \geqslant \max(\text{lmesh}(P), \text{lmesh}(Q)).$$

Remark. Looking at the formulation of Theorem 6 one can suspect that a stronger inequality $\text{lmesh}(P * Q) \geqslant \text{lmesh}(P)\,\text{lmesh}(Q)$ might hold. But this turns out to be false already for quadratic polynomials.

Remark. Notice that Theorem 6 does not follow from Proposition 5 (contrary to the case of constant coefficients) since not all (finite) multiplier sequences considered as differential operators can be represented as the product of operators of the form $\lambda p + x p'$. On the other hand, Proposition 5 follows from Theorem 6 since for any given positive integer k one can represent the action of the operator $\lambda + x \frac{d}{dx}$ on polynomials of degree k as the Schur–Szegő composition with an appropriate polynomial, see Section 2. Note also that the Schur–Szegő product of two polynomials of which one has all real roots and the other all real roots of the same sign has all real roots.

2. Proofs

In the proofs of Theorems 2 and 6 we will use the following three facts collected in a lemma below.

Lemma 7.

(i) *Given two real polynomials f and g of the same degree one has that the pencil $cf(x) + dg(x)$ consists of hyperbolic polynomials if and only if f and g has all real and (non-strictly) interlacing roots. (This is known under the name Obreschkov's theorem, see [4] although it has been (re)discovered many times by different authors in the past.)*

(ii) *Given two polynomials P and Q of the same degree satisfying the conditions that P and Q are hyperbolic and, additionally, all roots of Q are of the same sign, then their Schur–Szegő composition $(P * Q)$ is hyperbolic. (This is a special case of the well-known Malo–Schur–Szegő theorem, see e.g. [7].)*

(iii) *Take $P(x) = a(x + x_1)(x + x_2) \cdots \cdots (x + x_k)$, where $0 < x_1 < x_2 < \cdots < x_k$, and choose $\lambda > 1$. Then the zeros of $P(x)$ and $P(\lambda x)$ are interlacing if and only if $\lambda < \text{lmesh}(P)$. Analogously, the zeros of $P(x)$ and $P(x + \lambda)$ are interlacing if and only if $\lambda < \text{mesh}(P)$.*

Let us settle Theorem 2.

Proof of Theorem 2. Given a linear ordinary differential operator A with constant coefficients and of finite order suppose that there exists $P \in \mathbb{R}[x]$ with all real zeros such that $\text{mesh}(A(P)) < \text{mesh}(P)$. Choose λ satisfying the inequalities $0 \leqslant \text{mesh}(A(P)) < \lambda < \text{mesh}(P)$. Then by (iii) the zeros of the polynomials $A(P)(x)$ and $A(P)(x + \lambda)$ are not interlacing. Using (i) we get that there exist $c, d \in \mathbb{R}$ such that the polynomial $cA(P)(x) + dA(P)(x + \lambda)$ has a non-real zero. But $cA(P)(x) + dA(P)(x + \lambda) = A(L)(x)$, where $L(x) = cP(x) + dP(x + \lambda)$. Since $\lambda < \text{mesh}(P)$ the zeros of $P(x)$ and $P(x + \lambda)$ are interlacing, and by (i) we conclude that all zeros of L are real. But, this implies that all zeros of $A(L)$ should be real as well. This contradiction finishes the proof. $\square$

We now settle Theorem 6.

Proof of Theorem 6. Given P and Q two polynomials of the same degree with all real roots of the same sign consider $S(x) := (P * Q)(x)$. Assume that $\text{lmesh}(S) < \text{lmesh}(P)$ and choose λ such that $1 \leqslant \text{lmesh}(S) < \lambda < \text{lmesh}(P)$. By (iii) since $\text{lmesh}(S) < \lambda$ the zeros of polynomials $S(x)$ and $S(\lambda x)$ are not interlacing. By (i) there exist $c, d \in \mathbb{R}$ such that the polynomial $cS(x) + dS(\lambda x)$ has a non-real zero. We have

$$cS(x) + dS(\lambda x) = \sum_{j=0}^{k} \binom{k}{j}(ca_j + d\lambda^j a_j)b_j x^j.$$

Set $L(x) = \sum_{j=0}^{j} \binom{k}{j}(ca_j + d\lambda^j a_j)x^j = cP(x) + dP(\lambda x)$. Then one gets $cS(x) + dS(\lambda x) = (L * Q)(x)$. By (iii) the inequality $\lambda < \text{lmesh}(P)$ implies that the zeros of polynomials $P(x)$ and $P(\lambda x)$ are interlacing. Then by (i) all the zeros of $L(x)$ are real, and all the zeros of $Q(x)$ are real and of the same sign. Then by (ii) all the zeros of $cS(x) + dS(\lambda x)$ should be real. This contradiction finishes the proof. □

Let us finally deduce Proposition 5 from Theorem 6.

Proof of Proposition 5. One can easily check that for any two polynomials P and Q of the same degree the relation $(P + axP') * Q = (P * Q) + ax(P * Q)'$ holds. When $P(x) = (x + 1)^k$ one gets $P * Q = Q$ and

$$Q + axQ' = \big((x + 1)^{k-1}\big((1 + ak)x + 1\big)\big) * Q.$$

Therefore, for $a > 0$ the action of the differential operator $1 + ax\frac{d}{dx}$ on the polynomial Q coincides with its composition with a polynomial with all negative roots and the result follows. □

3. Final remarks

Remark 1. Theorem 6 shows that for any two finite multiplier sequences of the same length the logarithmic mesh of their composition is greater than or equal to the maximum of their logarithmic meshes. One can try to generalize this result to the case of usual (infinite) multiplier sequence. Given a multiplier sequence $\mathcal{A} = \{\alpha_n\}_{n=0}^{\infty}$ defines its logarithmic mesh as

$$\text{lmesh}(\mathcal{A}) = \inf_{k=0,\dots,\infty} \text{lmesh}(\mathcal{A}_k),$$

where $\mathcal{A}_k$ is the k-th truncation of $\mathcal{A}$, i.e. its initial finite segment of length $k + 1$. Then Theorem 6 immediately implies the following.

Corollary 1. *For any infinite multiplier sequences $\mathcal{A}$ and $\mathcal{B}$ one has*

$$\text{lmesh}(\mathcal{AB}) \geqslant \max\big(\text{lmesh}(\mathcal{A}), \text{lmesh}(\mathcal{B})\big).$$

Problem 1. Describe the class of multiplier sequences whose logarithmic mesh is strictly greater than 1.

Remark 2. Theorems 2 and 6 are examples of a statement that an appropriate version of mesh on the space of hyperbolic polynomials is preserved under the action of the corresponding class of hyperbolicity preservers. At the moment we have no idea what kind of mesh one should associate to an arbitrary HPO so that its action on hyperbolic polynomials will increase it for generic hyperbolic polynomials. Such a notion would be highly desirable.

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