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A Laplace transform certified reduced basis method; application to the heat equation and wave equation

Une méthode de bases réduites « certifiée » utilisant la transformation de Laplace ; Application à l'équation de la chaleur et à l'équation des ondes

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ABSTRACT

We present a certified reduced basis (RB) method for the heat equation and wave equation. The critical ingredients are certified RB approximation of the Laplace transform; the inverse Laplace transform to develop the time-domain RB output approximation and rigorous error bound; a (Butterworth) filter in time to effect the necessary "modal" truncation; RB eigenfunction decomposition and contour integration for Offline–Online decomposition. We present numerical results to demonstrate the accuracy and efficiency of the approach.

RÉSUMÉ

On introduit une méthode de bases réduites « certifiée » pour l'équation de la chaleur et pour l'équation des ondes. Les outils sont les suivants : approximation en bases réduites « certifiée » de la transformée de Laplace, transformée de Laplace inverse pour l'approximation de la sortie en bases réduites pour la variable temps, estimations d'erreurs rigoureuses, filtre en temps (de Butterworth) mettant en évidence la nécessité d'une troncature « *modale* », décomposition en fonctions propres en bases réduites, intégrale de contour pour la décomposition « *Offline–Online* ». On donne des résultats numériques pour montrer l'éfficacité et la précision de la méthode.

Version française abrégée

On considère une équation de la chaleur et une équation des ondes qui dépendent d'un paramètre μ, avec une forme sesquilinéaire m (de masse) et une forme sesquilinéaire a (de raideur) toutes deux affines par rapport au paramètre, voir (1). Une solution approchée de référence est supposée obtenue par la méthode de Galerkin utilisant des éléments finis, $u^{\mathcal{N}}(t;\mu) \in X^{\mathcal{N}}$. La sortie est donnée par une fonctionnelle filtre du champ (Butterworth), voir (2). On reformule le problème en termes de transformée de Laplace : pour une fréquence ω, la fonction $\hat{u}^{\mathcal{N}}(\omega;\mu)$ satisfait $\forall v \in X^{\mathcal{N}}$ $\mathcal{A}(\hat{u}^{\mathcal{N}}(\omega;\mu), v; \omega; \mu) = \hat{g}(i\omega) f(v)$. Dans cette relation, $\mathcal{A}(w, v; \omega; \mu) \equiv \mathcal{G}(\omega) a(w, v; \mu) + \mathcal{H}(\omega) m(w, v; \mu)$, avec $\mathcal{G}(\omega) = 1$ (resp., $1 + i\omega\epsilon$), et $\mathcal{H} = i\omega$ (resp., $-\omega^2$) dans les cas parabolique (resp. hyperbolique) ; $g(t) = (1/6)t^3 e^{-t}$ est la fonctionnelle de contrôle ; $f(v)$ est la donnée. La sortie peut être réécrit comme dans (3).

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On introduit ensuite une hiérarchie d'espaces X_N de dimension N pour les approximations en bases réduites, ces espaces sont engendrés par un algorithme « glouton ». Étant donnés une fréquence ω et un paramètre μ l'approximation en bases réduites vérifie $\forall v \in X_N$ $\mathcal{A}(\hat{u}_N(\omega;\mu), v; \omega; \mu) = \hat{g}(i\omega) f(v)$. La sortie en bases réduites est alors donnée par (4). On construit, finalement, un estimateur d'erreur (5). Comme on l'énonce dans la Proposition 2.1, la quantité $\Delta_N^s(t;\mu)$ donne une estimation de l'erreur entre l'approximation en bases réduites (4) et la solution de référence en éléments finis (3).

La sortie en bases réduites peut être écrite sous la forme (6). L'intégrale $J_n(\mu)$ peut être évaluée par la méthode des résidus sous la forme (7). Dans cette expression, $\chi_N^{(n)}(\mu)$ et $\lambda_N^{(n)}(\mu)$ représentent respectivement les fonctions propres et les valeurs propres d'un problème aux valeurs propres en bases réduites. Des procédés classiques en bases réduites « *Offline–Online* » peuvent alors être utilisés, donnant une complexité algorithmique d'ordre $O(N^3 + Nn_{\mathrm{f}})$. Des stratégies « *Offline–Online* » peuvent être aussi utilisées dans le calcul de l'estimation d'erreur. La complexité algorithmique « *Online* » est encore indépendante de la dimension $\mathcal{N}$ de l'espace des éléments finis $X^{\mathcal{N}}$ utilisé dans le calcul de la solution de référence.

Les résultats numériques présentés Fig. 1 illustrent l'efficacité et la précision de la technique dans le cas parabolique (a), et dans la cas hyperbolique (b) : l'estimation d'erreur est petite (l'erreur réelle est encore plus petite), le coût du calcul est réduit d'un ordre de grandeur si on la compere à celui de la solution de référence.

1. Introduction

Current reduced basis (RB) treatment of parabolic equations [3] is quite effective, however RB treatment of hyperbolic equations suffers from pessimistic error bounds [11]. Here we introduce a different approach — based directly on continuous time rather than a temporal discretization — which takes advantage of the Laplace transform (LT) and inverse LT to provide sharper error bounds for evolution equations.

Several previous efforts inform our work; note here $\sigma (= \phi + i\omega)$ shall denote the LT variable. 1) Modal analyses consider expansions in eigenfunctions to yield reduced dynamical systems [6]. In our work, we replace the expansion with the inverse LT, in which a filter provides the modal truncation; we replace the eigenfunctions with the RB approximation of the LT — for given frequency ω, a coercive or noncoercive *elliptic* PDE [9,12]. Note we cannot provide rigorous error bounds for RB approximations of eigenproblems [4]; however, we can provide rigorous error bounds for the RB approximation of the LT. 2) The Krylov/moment-matching techniques of [1,5] and Fourier approaches of [13] consider the LT to identify a reduced-order space; this reduced-order space then serves in subsequent Galerkin projection in the time domain. In our work, we explicitly invoke the inverse LT to construct our RB approximation in the time domain; this permits the direct incorporation of the rigorous RB (elliptic) error bounds into the parabolic and hyperbolic context. 3) The papers [10,7] consider application of the LT/inverse LT to finite element (FE) semi-discretizations for the purpose of parallel implementation. In our work, we accelerate the FE procedure by RB treatment of the LT; our error bounds provide the necessary certification. (We also address pole-related issues through Online exact integration.)

We introduce a spatial domain $\Omega \in \mathbb{R}^2$ with boundary $\partial\Omega$; we denote the Dirichlet portion of the boundary by $\partial\Omega^D$. We introduce the complex Hilbert spaces $L^2(\Omega) \equiv \{\int_\Omega |v|^2 < \infty\}$, $H^1(\Omega) \equiv \{v \in L^2(\Omega) \mid |\nabla v| \in L^2(\Omega)\}$, and $X = \{v \in H^1(\Omega) \mid v|_{\partial\Omega^D} = 0\}$. Here $|v| = \sqrt{vv^*}$ denotes modulus and * denotes complex conjugate. We associate to $L^2(\Omega)$ the inner product $(w, v) \equiv \int_\Omega wv^*$ and norm $\|w\| \equiv \sqrt{(w,w)}$ and to X the inner product $(w,v)_X \equiv \int_\Omega \nabla w \cdot \nabla v^*$ and induced norm $\|w\|_X \equiv \sqrt{(w,w)_X}$.

We introduce a real parameter μ which resides in a closed bounded parameter domain $\mathcal{D} \in \mathbb{R}^P$. We then define parametrized sesquilinear forms $m(\cdot,\cdot;\mu): X \times X \to \mathbb{C}$ ("mass") and $a(\cdot,\cdot;\mu): X \times X \to \mathbb{C}$ ("stiffness"); for all $\mu \in \mathcal{D}$, $m(w,w;\mu)$ and $a(w,w;\mu)$ must be *real* for all $w \in X^{\mathcal{N}}$. We assume that m (resp., a) is symmetric and furthermore continuous and coercive with respect to $L^2(\Omega)$ (resp., X). We also define antilinear bounded forms $f: X \to \mathbb{C}$ (data) and $\ell: X \to \mathbb{C}$ (output). Finally, we suppose that our bilinear forms m and a are "affine in parameter" such that

$$m(w,v;\mu) = \sum_{q=1}^{Q_m} \Theta_m^q(\mu) m^q(w,v), \qquad a(w,v;\mu) = \sum_{q=1}^{Q_a} \Theta_a^q(\mu) a^q(w,v); \tag{1}$$

here the Θ_m^q (resp., Θ_a^q) $: \mathcal{D} \to \mathbb{R}$, $1 \leqslant q \leqslant Q_m$ (resp., Q_a), are μ-dependent coefficient functions, and the m^q (resp., a^q), $1 \leqslant q \leqslant Q_m$ (resp., Q_a), are μ-independent sesquilinear forms.

We now define a "truth" finite element (FE) approximation space: a standard $\mathbb{P}_2$ polynomial FE approximation space $X^{\mathcal{N}} \subset X$ of dimension $\mathcal{N}$. Our finite element space $X^{\mathcal{N}}$ shall inherit the inner product and norm associated to X. We further define the dual space $(X^{\mathcal{N}})'$ and associated dual norm $\|\xi\|_{(X^{\mathcal{N}})'} = \sup_{v \in X^{\mathcal{N}}} |\xi(v)|/\|v\|_X$. We also introduce stability constants $\alpha^{\mathcal{N}}(\mu) = \inf_{v \in X^{\mathcal{N}}} a(v,v;\mu)/\|v\|_X^2$ and $\kappa^{\mathcal{N}}(\mu) = \inf_{v \in X^{\mathcal{N}}} a(v,v;\mu)/m(v,v;\mu)$.

The parabolic problem reads: Given any $\mu \in \mathcal{D}$, find (the real part of) $u^{\mathcal{N}}(t;\mu) \in X^{\mathcal{N}}$ such that $m(u_t^{\mathcal{N}}(t;\mu), v;\mu) + a(u^{\mathcal{N}}(t;\mu); v;\mu) = g(t) f(v)$, $\forall v \in X^{\mathcal{N}}$, subject to initial condition $u^{\mathcal{N}}(t=0;\mu) = 0$ (for simplicity we consider only homogeneous initial conditions). We shall consider a particular "smooth-start" control function $g(t) = (1/6)t^3 e^{-t}$. The hyperbolic problem reads: Specify a (Rayleigh or viscous) damping coefficient, $\epsilon \in \mathbb{R}_+$; given any $\mu \in \mathcal{D}$, find (the real part of) $u^{\mathcal{N}}(t;\mu) \in X^{\mathcal{N}}$ such that $m(u_{tt}^{\mathcal{N}}(t;\mu), v;\mu) + \epsilon a(u_t^{\mathcal{N}}(t;\mu), v;\mu) + a(u^{\mathcal{N}}(t;\mu); v;\mu) = g(t) f(v)$, $\forall v \in X^{\mathcal{N}}$, subject to initial conditions $u^{\mathcal{N}}(t=0;\mu) = (u_t^{\mathcal{N}})(t=0;\mu) = 0$. Our output of interest (both for the parabolic and hyperbolic cases) is then given by

$$s^{\mathcal{N}}(t;\mu) = \int_0^t B(t-t')\ell\big(u^{\mathcal{N}}(t';\mu)\big)\,\mathrm{d}t', \tag{2}$$

where B is the standard causal Butterworth filter of order n_f and cut-off frequency ω_f. Note the truth output is *defined* by (2) and hence is explicitly filtered: the modeler must select n_f and ω_f.

We now state the parabolic and hyperbolic problems in terms of the LT and inverse LT. (Note that we may consider here only Linear-Time-Invariant operators/forms.) We introduce a "combined" parameter $\tilde{\mu} \equiv (\omega;\mu)$ which resides in $\mathbb{R} \times \mathcal{D} \equiv \widetilde{\mathcal{D}}_\infty$. We next define a generalized Helmholtz problem: Given $(\omega;\mu) \in \widetilde{\mathcal{D}}_\infty$, $\hat{u}^{\mathcal{N}}(\omega;\mu) \in X^{\mathcal{N}}$ satisfies $\mathcal{A}(\hat{u}^{\mathcal{N}}(\omega;\mu), v;\omega;\mu) = \hat{g}(i\omega)\mathcal{F}(v)$, $\forall v \in X^{\mathcal{N}}$, where $\mathcal{A}(w,v;\omega;\mu) \equiv \mathcal{G}(\omega)a(w,v;\mu) + \mathcal{H}(\omega)m(w,v;\mu)$, and $\mathcal{F}(v) \equiv f(v)$. (In the case of non-zero initial conditions there will be additional terms in $\mathcal{F}$.) Here $\hat{g}$ is the LT of $g(t)$: $\hat{g}(\sigma) = 1/(\sigma+1)^4$. Note that, thanks to (1), $\mathcal{A}(w,v;\omega;\mu)$ admits an affine expansion in the "combined" parameter $(\omega;\mu)$. We specify $\mathcal{H}:\mathbb{R} \to \mathbb{C}$ and $\mathcal{G}:\mathbb{R} \to \mathbb{C}$: in the parabolic case, $\mathcal{G}(\omega) = 1$, $\mathcal{H}(\omega) = i\omega$; in the hyperbolic case, $\mathcal{G}(\omega) = 1 + i\omega\epsilon$, $\mathcal{H}(\omega) = -\omega^2$.

We now invoke the LT convolution property [2] and the inverse LT to express our output (2) as

$$s^{\mathcal{N}}(t;\mu) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \widehat{B}(i\omega)\ell\big(\hat{u}^{\mathcal{N}}(\omega;\mu)\big)e^{i\omega t}\,\mathrm{d}\omega. \tag{3}$$

Here $\widehat{B}$ is the LT of the Butterworth filter: for $\sigma \in \mathbb{C}$, $\widehat{B}(\sigma) = \omega_\mathrm{f}\prod_{k=1}^{n_\mathrm{f}}(\sigma - \sigma_\mathrm{f}^k)^{-1}$, where the $\sigma_\mathrm{f}^k = \omega_\mathrm{f}\exp(\frac{\pi i}{2} + \frac{\pi i}{2n_\mathrm{f}}) \times \exp(\frac{(k-1)\pi i}{n_\mathrm{f}})$, $1 \leqslant k \leqslant n_\mathrm{f}$, are the Butterworth poles.

In our inverse LT path we have chosen the real part, ϕ, to be zero. In fact, a non-zero shift ϕ can be gainfully exploited. In the parabolic case, we may choose negative ϕ (though still to the right of the eigenvalues of the differential operator and the poles of the filter) to obtain *decaying* error bounds. In the hyperbolic case, we may choose a positive ϕ in order to set $\epsilon = 0$ — no dissipation; we then obtain error bounds which grown linearly in t. We treat these cases in future work.

2. Reduced basis method

The reduced basis approximation shall be developed over the "combined" parameter $\tilde{\mu} \equiv (\omega;\mu)$; we shall also need the restricted parameter domain $\widetilde{\mathcal{D}} \equiv [0,\overline{\omega}] \times \mathcal{D}$ for some prescribed $\overline{\omega} > \omega_\mathrm{f}$ (note that under our assumptions $\hat{u}^{\mathcal{N}}(-\omega;\mu) = (\hat{u}^{\mathcal{N}}(\omega;\mu))^*$). We first introduce the RB approximation spaces [9] relevant to both the parabolic and hyperbolic cases. We identify $N_{\max}$ hierarchical RB approximation spaces X_N, $1 \leqslant N \leqslant N_{\max}$; here X_N is of dimension N. These "Lagrange" [8,9] RB spaces may be expressed as $X_N = \mathrm{span}\{\zeta_i,\ i = 1,\dots,N\}$, $1 \leqslant N \leqslant N_{\max}$, where the ζ_i, $1 \leqslant i \leqslant N_{\max}$, are $(\,,\,)_X$-orthonormalized snapshots $\hat{u}^{\mathcal{N}}(\tilde{\mu}^i_{\mathrm{Greedy}})$, $1 \leqslant i \leqslant N_{\max}$. The sample points $\tilde{\mu}^i_{\mathrm{Greedy}} \in \widetilde{\mathcal{D}}$, $1 \leqslant i \leqslant N_{\max}$, at which the snapshots are computed are selected by a Greedy procedure [12,9]. We might also consider a mixed approach with POD in ω [13] and Greedy in μ analogous to the time-domain scheme of [3].

The RB approximation for the LT is then given by Galerkin projection: Given $(\omega;\mu) \in \widetilde{\mathcal{D}}_\infty$, find $\hat{u}_N(\omega;\mu) \in X_N$ such that $\mathcal{A}(\hat{u}_N(\omega;\mu), v;\omega;\mu) = \hat{g}(i\omega)\mathcal{F}(v)$, $\forall v \in X_N$. Then, given t and μ, we evaluate the RB output as

$$s_N(t;\mu) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \widehat{B}(i\omega)\ell\big(\hat{u}_N(\omega;\mu)\big)e^{i\omega t}\,\mathrm{d}\omega. \tag{4}$$

For the appropriate choice of $\mathcal{G}$ and $\mathcal{H}$ this formulation applies to both the parabolic and hyperbolic cases. Galerkin projection chooses a good linear combination of the snapshots.

We next introduce the output error estimator for given time t and μ as

$$\Delta_N^s(t;\mu) \equiv \frac{\|\ell\|_{(X^{\mathcal{N}})'}}{2\pi\alpha_{\mathrm{LB}}^{\mathcal{N}}(\mu)\eta(\mu)}\left(\int_{-\infty}^{\infty}\big|\hat{g}(i\omega)\big|^2\,\mathrm{d}\omega\right)^{\frac{1}{2}}\left(\int_{-\infty}^{\infty}\big|\widehat{B}(i\omega)\big|^2\big\|\widehat{R}(\omega;\mu)\big\|_X^2\,\mathrm{d}\omega\right)^{\frac{1}{2}}, \tag{5}$$

where $\widehat{R}$ is the Riesz representation of the residual, $(\widehat{R}(\omega;\mu), v)_X = \mathcal{F}(v) - \hat{g}(i\omega)^{-1}\mathcal{A}(\hat{u}_N(\omega;\mu), v;\omega;\mu)$, $\forall v \in X^{\mathcal{N}}$, and $\alpha_{\mathrm{LB}}^{\mathcal{N}}$ is a lower bound for $\alpha^{\mathcal{N}}$ (provided by the Offline–Online SCM [9]). In the parabolic case $\eta(\mu) = 1$; in the hyperbolic case $\eta(\mu) = \tau(\epsilon(\kappa_{\mathrm{LB}}^{\mathcal{N}}(\mu))^{1/2})$, where $\tau(z) \equiv z(-z/2 + \sqrt{1 + z^2/4})$ and $\kappa_{\mathrm{LB}}^{\mathcal{N}}(\mu)$ is a lower bound for $\kappa^{\mathcal{N}}(\mu)$ (constructed by variants on the SCM). We can then state

Proposition 2.1. *For any $t > 0$ and $\mu \in \mathcal{D}$, we obtain $|s^{\mathcal{N}}(t;\mu) - s_N(t;\mu)| \leqslant \Delta_N^s(t;\mu)$.*

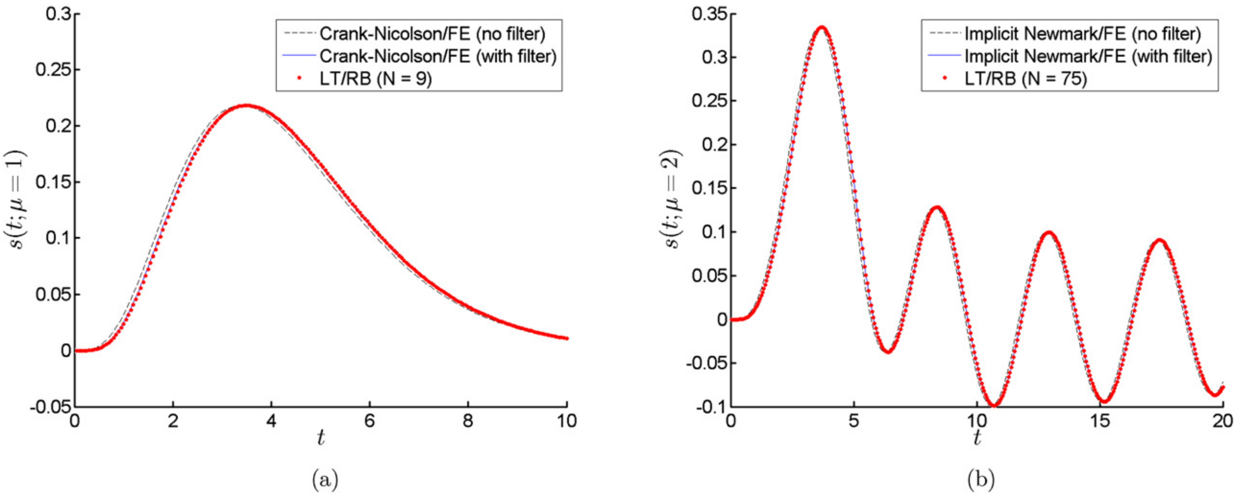

Fig. 1. Comparison of filtered ($\omega_{\mathrm{f}} = 60$, $n_{\mathrm{f}} = 10$) and unfiltered ($\omega_{\mathrm{f}} = \infty$) FE solutions with the (filtered) LT RB solution for the (a) parabolic problem, and (b) hyperbolic problem.

Although we only "train" the RB approximation over the finite interval $[-\overline{\omega}, \overline{\omega}]$, we define the RB approximation for all frequencies; this will be important in order to apply contour integration. The inaccuracy of the RB approximation for higher frequency is not of concern: the Butterworth filter severely attenuates these frequencies; and we are assured that the RB residual remains bounded thanks to stability.

It remains to develop a computational procedure for the RB approximation and error bound. We recall the Offline–Online RB strategy [9]: we admit significant Offline effort in exchange for greatly reduced cost in the Online stage — in which we aim to provide very rapid ("real-time") response for each new query $t, \mu \to s_N(t;\mu)$, $\Delta_N^s(t;\mu)$. We first introduce an RB eigensystem: given μ, $a(\chi_N(\mu), v; \mu) = \lambda_N(\mu) m(\chi_N(\mu), v; \mu)$, $\forall v \in X_N$, with eigenpairs $(\chi_N^{(n)}(\mu), \lambda_N^{(n)}(\mu)) \in (X_N, \mathbb{R})$, $1 \leqslant n \leqslant N$.

We may then write the RB output as

$$s_N(t;\mu) = \sum_{n=1}^{N} \ell\big(\chi_N^{(n)}(\mu)\big) f\big(\chi_N^{(n)}(\mu)\big) J_n(\mu), \tag{6}$$

where $J_n(\mu) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \widehat{B}(i\omega)\hat{g}(i\omega)\mathbb{A}_n(i\omega;\mu)e^{i\omega t}\,\mathrm{d}\omega$; here $\mathbb{A}_n(\sigma;\mu) = 1/(\lambda_N^{(n)}(\mu) + \sigma)$ and $\mathbb{A}_n(\sigma;\mu) = 1/((1+\epsilon\sigma)\lambda_N^{(n)}(\mu) + \sigma^2)$ in the parabolic and hyperbolic cases, respectively. The integral is readily evaluated by residues to yield (assuming distinct poles), for the hyperbolic problem,

$$\begin{aligned} J_n = {} & e^{\rho_+^{(n)}t}\big(\rho_+^{(n)} - \rho_-^{(n)}\big)^{-1}\widehat{B}\big(\rho_+^{(n)}\big)\hat{g}\big(\rho_+^{(n)}\big) + e^{\rho_-^{(n)}t}\big(\rho_-^{(n)} - \rho_+^{(n)}\big)^{-1}\widehat{B}\big(\rho_-^{(n)}\big)\hat{g}\big(\rho_-^{(n)}\big) \\ & + \sum_{k=1}^{n_{\mathrm{f}}} e^{\sigma_{\mathrm{f}}^k t}\big((1+\epsilon\sigma_{\mathrm{f}}^k)\lambda_N^{(n)} + \sigma^2\big)^{-1}\widehat{B}_k\big(\sigma_{\mathrm{f}}^k\big)\hat{g}\big(\sigma_{\mathrm{f}}^k\big) + \frac{1}{6}\frac{\mathrm{d}^3}{\mathrm{d}\sigma^3}\big\{e^{\sigma t}\big((1+\epsilon\sigma)\lambda_N^{(n)} + \sigma^2\big)^{-1}\widehat{B}(\sigma)\big\}\Big|_{\sigma=-1}, \end{aligned} \tag{7}$$

where $\rho_\pm^{(n)}(\mu) = -\epsilon\lambda_N^{(n)}(\mu)/2 \pm \sqrt{-\lambda_N^{(n)}(\mu) + (\epsilon\lambda_N^{(n)}(\mu)/2)^2}$; the parabolic case is similar. We observe the connection to modal approaches. In the Online stage we may assemble and solve the RB eigenproblem in $O(N^3)$ operations and form the $f(\chi_N^n)$, $\ell(\chi_N^n)$, $1 \leqslant n \leqslant N$, in $O(N^2)$ FLOPs; we then evaluate our output (6), (7) in $O(Nn_{\mathrm{f}})$ FLOPs per t requested. This result relies on standard RB Offline–Online procedures now supplemented with the RB eigenfunction representation and contour integration.

The Offline–Online approach for the error bound, $\Delta_N^s(t;\mu)$ of (5), is more involved, and the details are relegated to a future publication. The essential components are the standard RB Offline–Online decomposition, the RB eigenfunction expansion, and contour integration by residues. The complexity of the Online stage for the error bound is an additional $O(N^2(Q_m + Q_a)^2 + N^2 n_{\mathrm{f}})$.

We now consider two model problems, one parabolic and one hyperbolic, both posed on the same domain and over the same $\mathbb{P}_2$ FE truth approximation space of dimension $\mathcal{N} = 9989$. Let Ω be the unit square in $\mathbb{R}^2$; let Ω_1 denote the 0.5×0.5 square centered at $(0.5, 0.5)$, and define $\Omega_2 \equiv \Omega \setminus \Omega_1$. The boundary conditions are $\partial u/\partial n = 0$ on the top and bottom boundaries, $\partial u/\partial n = g(t)$ on the left boundary $\partial\Omega_{\mathrm{left}}$, and $u = 0$ on the right boundary $\partial\Omega^D$. The bilinear forms are $a(v, w;\mu) \equiv \int_{\Omega_1} \nabla v \cdot \nabla w + \mu\int_{\Omega_2} \nabla v \cdot \nabla w$, $m(v, w) \equiv \int_\Omega vw$, and $a(v, w;\mu) \equiv \int_\Omega \nabla v \cdot \nabla w$, $m(v, w) = \int_{\Omega_1} vw + \mu\int_{\Omega_2} vw$, for the parabolic and hyperbolic problems, respectively; the linear forms are $f(v) = \int_{\partial\Omega_{\mathrm{left}}} v$ and $\ell(v) = f(v)$; the filter is specified by $n_f = 10$, $\omega_f = 60$. In the hyperbolic case we choose a damping coefficient of $\epsilon = 2E - 2$ (and employ the lower bound $\kappa_{LB}^{\mathcal{N}}(\mu) = \kappa^{\mathcal{N}}(1)/\mu$). We consider the parameter domain $\mathcal{D} \equiv [1, 4]$.

We generate a reduced basis with $\overline{\omega} = 120$ to satisfy an error bound tolerance $\varepsilon_{tol} = 1E - 2$ over a Greedy training set of size $n_{train} = 10\,000$; we require $N_{max} = 10$ for the parabolic case and $N_{max} = 85$ for the hyperbolic case. We show in Fig. 1(a) the RB results for the parabolic case for $\mu = 1$ and $N = 9$; the RB error bound is $\Delta_N^s(t; \mu) = 6.2E - 3$ for all time t. The Online RB (output and error bound) is 50 times faster than a Crank–Nicolson ($\Delta t = 0.1$) FE truth. We show in Fig. 1(b) the RB results of the hyperbolic case for $\mu = 2$ and $N = 75$; the RB error bound is $\Delta_N^s(t; \mu) = 5.7E - 3$ for all time t. The Online RB is 40 times faster than a second-order Implicit Newmark ($\Delta t = 0.05$) FE truth. We introduce a temporal discretization of the truth to more meaningfully compare computational cost.

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