C. R. Acad. Sci. Paris, Ser. I 349 (2011) 603–606

Contents lists available at ScienceDirect

C. R. Acad. Sci. Paris, Ser. I

www.sciencedirect.com

Number Theory/Algebraic Geometry

Squareful points of bounded height

Points puissants de hauteur bornée

Karl Van Valckenborgh

K.U. Leuven, Department of Mathematics, Celestijnenlaan 200B, 3001 Leuven, Belgium

ARTICLE INFO

Article history:
Received 18 March 2011
Accepted 2 May 2011
Available online 1 June 2011

Presented by the Editorial Board

ABSTRACT

Let $n \geqslant 5$. In this Note, we explain how to determine the asymptotic behaviour of the size of the set of rational points $(a_0 : \ldots : a_n) \in \mathbf{P}^n(\mathbf{Q})$ (where $a_0, \ldots, a_n \in \mathbf{Z}$ and $\gcd(a_0, \ldots, a_n) = 1$) of bounded height $\max_{i=0,\ldots,n} |a_i| \leqslant B$ on the hyperplane $\sum_{i=0}^n X_i = 0$ such that a_i is squareful for each $i \in \{0, \ldots, n\}$ as B goes to infinity. (An integer a is called *squareful* if the exponent of each prime divisor of a is at least two.) The main tool we will use, is the (classical) Hardy–Littlewood circle method.

RÉSUMÉ

Soit $n \geqslant 5$. Dans cette Note, nous expliquerons comment on peut déterminer le comportement asymptotique du nombre de points rationnels $(a_0 : \ldots : a_n) \in \mathbf{P}^n(\mathbf{Q})$ (avec $a_0, \ldots, a_n \in \mathbf{Z}$ et $\mathrm{pgcd}(a_0, \ldots, a_n) = 1$) de hauteur bornée $\max_{i=0,\ldots,n} |a_i| \leqslant B$ sur l'hyperplan $\sum_{i=0}^n X_i = 0$ tels que a_i est un entier puissant pour chaque $i \in \{0, \ldots, n\}$, lorsque B tend vers l'infini. (Un entier a est appelé *puissant* si pour chaque nombre premier p divisant a, on a que p^2 aussi divise a.) La méthode principale qu'on utilise ici est la méthode du cercle de Hardy–Littlewood (classique).

1. Introduction

The problem we consider can be related to an unsolved question Campana posed when examining rational points on orbifolds. A good overview and setup of the Campana program is given for example in [1,4] or [2].

In the easiest configuration, an orbifold consists of a $\mathbf{Q}$-rational divisor

$$\Delta = \sum_{i=1}^N \left(1 - \frac{1}{m_i}\right) \cdot [P_i]$$

on the projective line $\mathbf{P}^1$ over $\mathbf{Q}$, where $P_i \in \mathbf{P}^1(\mathbf{Q})$ and $m_i \in \{2, 3, \ldots\} \cup \{\infty\}$ for every $i \in \{1, \ldots, N\}$. We denote such an orbifold by $(\mathbf{P}^1, \Delta)$. Considering rational points P on the projective line which 'behave well' with respect to Δ (namely such that for each prime number p and each rational point P_i (supporting Δ) which intersects P above p, it holds that this intersection number is at least m_i; for more details see e.g. [1, Section 2]), it follows from the Campana program that for the specific case where $\Delta = 1/2 \cdot [0] + 1/2 \cdot [1] + 1/2 \cdot [\infty]$ it is predicted (but yet to be proved) that the size of the set of

E-mail address: karl.vanvalckenborgh@wis.kuleuven.be.

1631-073X/$ – see front matter
doi:10.1016/j.crma.2011.05.001

points $(a_0 : a_1) \in \mathbf{P}^1(\mathbf{Q})$ (where $a_0, a_1 \in \mathbf{Z}$ and $\gcd(a_0, a_1) = 1$) such that a_0, a_1 and $a_0 + a_1 = a_2$ are squareful integers and $\max\{|a_0|, |a_1|, |a_2|\} \leqslant B$ (we denote this set by $(\mathbf{P}^1, \Delta)(\mathbf{Q})_{\leqslant B}$) will behave asymptotically as $C \cdot B^{1/2}$ for some constant $C > 0$ as B goes to infinity.

Generalizing to higher dimension and thus adding more variables, it is reasonable to expect that the size of the set $\{(a_0 : \ldots : a_{n-1}) \in \mathbf{P}^{n-1}(\mathbf{Q})$ (where $a_0, \ldots, a_{n-1} \in \mathbf{Z}$ and $\gcd(a_0, \ldots, a_{n-1}) = 1$) for which $a_0, \ldots, a_{n-1}$ and $\sum_{i=0}^{n-1} a_i = a_n$ are squareful integers and $\max_{i=0,\ldots,n} |a_i| \leqslant B$ (analogously denoted as $(\mathbf{P}^{n-1}, \Delta)(\mathbf{Q})_{\leqslant B}$) will behave asymptotically as $C \cdot B^{(n-1)/2}$ for some constant $C > 0$ as B goes to infinity.

This is exactly what we will prove provided that $n \geqslant 5$. We will first determine an asymptotic formula, using the Hardy–Littlewood circle method, for the size of the set $M_{\underline{a},t}(B)$ of integral solutions $(x_0, \ldots, x_n, y_0, \ldots, y_n) \in \mathbf{Z}_0^{2n+2}$ (here, $\mathbf{Z}_0 = \mathbf{Z} \setminus \{0\}$) of the equation $\sum_{i=0}^n a_i x_i^2 y_i^3 = t$ that satisfy $\max_{i=0,\ldots,n} |a_i x_i^2 y_i^3| \leqslant B$ and y_i squarefree for each $i \in \{0, \ldots, n\}$, where $a_0, \ldots, a_n, t \in \mathbf{Z}$ are fixed, $\gcd(a_0, \ldots, a_n) = 1$ and $\prod_{i=0}^n a_i \neq 0$ (see Theorem 3.1). Next, we will give describe the asymptotic behaviour of the cardinality of the set $M_{\underline{1},0}(B)$ with the additional condition $\gcd(x_i y_i,\ i = 0, \ldots, n) = 1$ on the solutions; we denote this set by $M(B)$. Here, we will explain how we can bring this gcd condition into account using some kind of Möbius inversion (Theorem 3.2). Finally, since a squareful integer can be written 'uniquely' as $x^2 y^3$ where y is squarefree (recall that y squarefree is equivalent to the fact that $\mu^2(|y|) = 1$ where $\mu(\cdot)$ is the Möbius function; furthermore, this representation of a squareful integer is unique up to the sign of x), the points with nonzero coordinates in $(\mathbf{P}^{n-1}, \Delta)(\mathbf{Q})_{\leqslant B}$, denoted by $(\mathbf{P}^{n-1}, \Delta)(\mathbf{Q})^+_{\leqslant B}$, corresponds to the set

$$\Big\{\big(x_0^2 y_0^3 : \ldots : x_n^2 y_n^3\big) \in H(\mathbf{Q}) \,\Big|\, x_i, y_i \in \mathbf{Z}_0,\ y_i \text{ squarefree},\ \gcd(x_i y_i,\ i = 0, \ldots, n) = 1 \text{ and } \max_{i=0,\ldots,n} \big|x_i^2 y_i^3\big| \leqslant B\Big\} \tag{1}$$

where $H \subset \mathbf{P}^n$ denotes the hyperplane defined by the equation $X_0 + \cdots + X_n = 0$. (We will identify these two sets from now on.) Hence,

$$\#\big(\mathbf{P}^{n-1}, \Delta\big)(\mathbf{Q})^+_{\leqslant B} = \frac{1}{2^{n+2}} \# M(B)$$

keeping in mind that in $(\mathbf{P}^{n-1}, \Delta)(\mathbf{Q})^+_{\leqslant B}$ the x_i are defined up to sign and, since we are considering projective points, the $n+1$-tuple $(y_0, \ldots, y_n)$ is also defined up to sign (as $n+1$-tuple). From this, it follows that the asymptotic formula for $\# M(B)$ will induce an asymptotic formula for $\#(\mathbf{P}^{n-1}, \Delta)(\mathbf{Q})^+_{\leqslant B}$ (and hence also for $\#(\mathbf{P}^{n-1}, \Delta)(\mathbf{Q})_{\leqslant B}$, since looking at points with nonzero coordinates is simply an open condition and does not change the asymptotic formula).

2. Calculating $\# M_{\underline{a},t}(B)$

First of all, let us fix the framework of the circle method.

Let T be $\mathbf{R}/\mathbf{Z}$. For $0 < \Delta \leqslant 1$, we define $\mathfrak{M}(\Delta, q, a)$ as the image in T of $\{\alpha \in \mathbf{R}\colon\ |\alpha - a/q| < B^{(\Delta-2)/2}\}$ with $a, q \in \mathbf{Z}$ and

$$\mathfrak{M}(\Delta) = \bigcup_{\substack{1 \leqslant a \leqslant q \leqslant B^{\Delta/2} \\ \gcd(a,q)=1}} \mathfrak{M}(\Delta, q, a)$$

called the union of the *major arcs* and $T \setminus \mathfrak{M}(\Delta) = \mathfrak{m}(\Delta)$ the union of the *minor arcs*. This definition is clearly dependent of the choice of Δ, which we will have to determine properly for this technique to work.

The circle method calculates $\# M_{\underline{a},t}(B)$ by integrating an exponential sum over T, namely

$$\# M_{\underline{a},t}(B) = \int_T \sum_{\substack{1 \leqslant |a_i x_i^2 y_i^3| \leqslant B \\ i=0,\ldots,n}} \left(\prod_{i=0}^n \mu^2\big(|y_i|\big)\right) e\big(\alpha f(\underline{x}, \underline{y})\big)\, \mathrm{d}\alpha,$$

where $f(\underline{x}, \underline{y}) = \sum_{i=0}^n a_i x_i^2 y_i^3 - t$. (From now on, $e(x) = \exp(2\pi i x)$ for $x \in \mathbf{R}$.) For the integrand of this integral, denoted by $E(\alpha)$, it holds that $E(\alpha) = e(-\alpha t) \prod_{i=0}^n S_i(\alpha)$ putting $S_i(\alpha) = \sum_{1 \leqslant |a_i x^2 y^3| \leqslant B} \mu^2(|y|) e(\alpha a_i x^2 y^3)$.

2.1. Major arcs

We will use the classical circle method, as described in detail in e.g. [5] or [3]. We can prove the following theorem:

Theorem 2.1. *For $n \geqslant 5$, it holds, some constant $\delta > 0$ and for $0 < \Delta < 1/15$, that*

$$\int_{\mathfrak{M}(\Delta)} E(\alpha)\, \mathrm{d}\alpha = C_{\underline{a},t} \cdot B^{(n-1)/2} + O\big(B^{(n-1)/2-\delta}\big) \quad \textit{with} \quad C_{\underline{a},t} = 2^{n+1} \sum_{(y_0,\ldots,y_n) \in \mathbf{Z}_0^{n+1}} \left(\prod_{i=0}^n \mu^2\big(|y_i|\big)\right) \frac{\mathfrak{S}_{\underline{y},\underline{a},t} \mathfrak{I}_{\underline{\varepsilon}}}{\prod_{i=0}^n |a_i y_i^3|^{1/2}}$$

where (*putting* $\varepsilon_i = \operatorname{sgn}(a_i y_i)$)

$$\mathfrak{S}_{\underline{y},\underline{a},t} = \sum_{q=1}^{\infty} \sum_{\substack{0<\frac{a}{q}\leqslant 1 \\ \gcd(a,q)=1}} q^{-(n+1)} \sum_{\underline{z}\in(\mathbf{Z}/q\mathbf{Z})^{n+1}} e\big(\big(af(\underline{z},\underline{y})\big)/q\big) \quad \textit{and} \quad \mathfrak{I}_{\underline{\varepsilon}} = \int_{-\infty}^{+\infty} \mathrm{d}\gamma \int_{[0,1]^{n+1}} e\left(\gamma \sum_{i=0}^{n} \varepsilon_i x_i^2\right) \mathrm{d}\underline{x}.$$

Our strategy to treat the integral is to first look at the equation $f(\underline{x},\underline{y}) = f_{\underline{y}}(\underline{x}) = 0$ with $\underline{y}$ fixed; afterwards we will take the sum over all admitted $\underline{y}$ (keeping in mind that each y_i has to be squarefree). So, fixing $\underline{y}$ and hence looking at the diagonal equation $f_{\underline{y}}(\underline{x}) = 0$, we can use the circle method in a well-known way to prove the following proposition:

Proposition 2.2. *For* $n \geqslant 5$, *it holds that*

$$\int_{\mathfrak{M}(\Delta)} E_{\underline{y}}(\alpha)\,\mathrm{d}\alpha = \frac{2^{n+1}\mathfrak{S}_{\underline{y},\underline{a},t}\mathfrak{I}_{\underline{\varepsilon}}}{\prod_{i=0}^{n}|a_i y_i^3|^{1/2}} B^{(n-1)/2} + O\left(\frac{B^{(n-2)/4}}{\operatorname{lcm}(y_0,\ldots,y_n)^{3/2}} + \frac{B^{(\Delta-2)(1-n)/4}}{\prod_{i=0}^{n}|a_i y_i^3|^{1/2}} + \frac{\sum_{i=0}^{n}|a_i y_i^3|^{1/2}}{\prod_{i=0}^{n}|a_i y_i^3|^{1/2}} B^{(n+5\Delta-2)/2}\right) \tag{2}$$

with $E_{\underline{y}}(\alpha) = \sum_{\substack{1/|a_i y_i^3|^{1/2}\leqslant|x_i|\leqslant B_{a_i y_i} \\ i=0,\ldots,n}} e(\alpha f_{\underline{y}}(\underline{x}))$.

Here, we first have to examine $E_{\underline{y}}(\alpha)$ for $\alpha \in \mathfrak{M}(\Delta,q,a)$ from which we then can derive the expression (2) by calculating the integral of $E_{\underline{y}}(\alpha)$ over $\mathfrak{M}(\Delta,q,a)$ and afterwards summing over all admitted a and q.

For this proposition to make sense, we have to check that the coefficient of the main term converges as B goes to infinity and determine Δ such that the error term is $O_{\underline{y}}(B^{(n-1)/2-\delta})$ for a $\delta > 0$. For the latter, choosing $0 < \Delta < 1/5$ suffices; for the coefficient on the other hand, the following lemma is needed:

Lemma 2.3. *We have* $|q^{-(n+1)} \sum_{\underline{z}\in(\mathbf{Z}/q\mathbf{Z})^{n+1}} e((af_{\underline{y}}(\underline{z}))/q)| \ll q^{-n/2} \cdot \frac{\prod_{i=0}^{n}|a_i y_i^3|^{1/2}}{\operatorname{lcm}(y_0,\ldots,y_n)^{3/2}}$.

This can be proved easily using some basic facts concerning generalised Gauss sums and implies that $\mathfrak{S}_{\underline{y},\underline{a},t}$ converges for $n \geqslant 5$.

To conclude the proof of Theorem 2.1, the coefficient also has to converge as B tends to infinity when summing over all admitted y_i (using the same lemma as before) and the error term has to be of the form $O(B^{(n-1)/2-\delta})$ for some $\delta > 0$; for the latter, we need that $0 < \Delta < 1/15$.

2.2. Minor arcs

For the minor arcs $\mathfrak{m}(\Delta)$, we do not fix $\underline{y}$ but examine the whole equation at once. We will explain the different steps needed to prove the following theorem:

Theorem 2.4. *For* $n \geqslant 5$, *we have* $\int_{\mathfrak{m}(\Delta)} E(\alpha)\,\mathrm{d}\alpha = O(B^{(n-1)/2-\delta})$ *for some* $\delta > 0$.

(Notice that we do not have to impose an extra condition on Δ.)

Using Hölder's inequality, we first of all see that

$$\left|\int_{\mathfrak{m}(\Delta)} E(\alpha)\,\mathrm{d}\alpha\right| \leqslant \sup_{\alpha\in\mathfrak{m}(\Delta)} \big(|S_0(\alpha)|\cdots|S_{n-4}(\alpha)|\big) \cdot \max_{j=n-3,\ldots,n} \int_0^1 |S_j(\alpha)|^4\,\mathrm{d}\alpha. \tag{3}$$

For the integral, we obtain following upper bound:

Lemma 2.5. *For any* $\varepsilon > 0$, *we have* $\int_0^1 |S_j(\alpha)|^4\,\mathrm{d}\alpha \ll_\varepsilon B^{1+\varepsilon}$.

The proof of this lemma essentially boils down in counting the number of solutions $(\underline{x},\underline{y}) \in \mathbf{Z}^7$ of $y_3^3(x_3^2 - x_4^2) = x_1^2 y_1^3 - x_2^2 y_2^3$ such that $1 \leqslant x_i < B^{1/2}/Y^{3/2}$, $Y < y_j \leqslant 2Y\}$ after applying Cauchy inequality. (Remark that this lemma implies that the equation $n_1 + n_2 = n_3 + n_4$, where n_i is squareful and $|n_i| \leqslant B$ for each $i \in \{1,2,3,4\}$, has $O(B^{1+\varepsilon})$ solutions.)

If we now focus on the other part in (3), namely on $\sup_{\alpha\in\mathfrak{m}(\Delta)}(|S_0(\alpha)|\cdots|S_{n-4}(\alpha)|)$, we see this contains at least two factors if $n \geqslant 5$. We may assume, after possibly renumbering the indices, that $|a_0| = \min_{i=0,\ldots,n}|a_i|$. Using a classical reasoning, often used when studying the integral over the minor arcs, called *Weyl's inequality* (this can be found in e.g. [3, Chapter 3]), it follows

Proposition 2.6. *We have* $|S_0(\alpha)| \ll |a_0|^{1/4+\varepsilon'} B^{1/2-\delta}$, *for a* $\delta > 0$ *and any* $\varepsilon' > 0$.

Combining the trivial upper bound $|S_i(\alpha)| \ll B^{1/2}/|a_i|^{1/2}$ for the other factors with Proposition 2.6 and Lemma 2.5 completes the proof of Theorem 2.4.

3. Towards the main problem

From Theorem 2.1 and Theorem 2.4, it follows that

Theorem 3.1. *For* $n \geqslant 5$, *we have* $\#M_{\underline{a},t}(B) = C_{\underline{a},t} \cdot B^{(n-1)/2} + O(B^{(n-1)/2-\delta})$, *for some* $\delta > 0$ *and* $C_{\underline{a},t}$ *as described in Theorem* 2.1.

We can now use this theorem to determine the size of the set $M(B)$ as defined in the introduction.

Theorem 3.2. *For* $n \geqslant 5$, *it holds that* $\#M(B) = C \cdot B^{(n-1)/2} + O(B^{(n-1)/2-\delta})$ *for some* $\delta, C > 0$. *An explicit description of* C *is given in* (4).

Since we already have a (similar) asymptotic formula for the size of the set $M_{\underline{1},0}(B)$ (but without the coprimality condition) the only problem still left to prove Theorem 3.2 is to see how the gcd condition $\gcd(x_i y_i,\ i = 0, \ldots, n) = 1$ comes in. Notice that this is not so trivial: the Möbius inversion we need here leads to divisibility conditions on both x_i and y_i which are rather tricky to handle. The key idea follows from the inclusion–exclusion principle. Denoting the set

$$\left\{ (\underline{x}, \underline{y}) \in \mathbf{Z}_0^{2n+2} \;\Big|\; \sum_{i=0}^{n} x_i^2 y_i^3 = 0,\ \max_{i=0,\ldots,n} \left|x_i^2 y_i^3\right| \leqslant B,\ e_i | x_i,\ f_i | y_i \text{ for all } i \in \{0, \ldots, n\} \right\}$$

(where $e_i, f_i \in \mathbf{N}$ and f_i (and of course y_i) squarefree for each i) by $N_{(\underline{e},\underline{f})}(B)$, we get $\#N_{(\underline{e},\underline{f})}(B) = \#M_{\underline{e^2 f^3},0}(B)$ and thus from Theorem 3.1 that $\#N_{(\underline{e},\underline{f})}(B) = C_{\underline{e^2 f^3},0} \cdot B^{(n-1)/2} + O(B^{(n-1)/2-\delta})$. Defining an adapted Möbius function $\mu : \mathbf{N}^{n+1} \times \mathbf{N}^{n+1} \to \mathbf{Z} : (\underline{e}, \underline{f}) \mapsto \mu(\underline{e}, \underline{f})$ such that

$$\#M(B) = \sum_{e=1}^{\infty} \sum_{\substack{(\underline{e},\underline{f}) \in \mathbf{N}^{2n+2} \\ e = \gcd(e_i f_i,\ i=0,\ldots,n)}} \mu(\underline{e}, \underline{f}) \cdot \#N_{(\underline{e},\underline{f})}(B),$$

we can then prove Theorem 3.2, with the (convergent) series C defined as

$$C = \sum_{e=1}^{\infty} \sum_{\substack{(\underline{e},\underline{f}) \in \mathbf{N}^{2n+2} \\ \gcd(e_i f_i,\ i=0,\ldots,n) = e}} \mu(\underline{e}, \underline{f}) \cdot C_{\underline{e^2 f^3},0}. \tag{4}$$

Notice that this is not so trivial: to do this, it is essential to notice that the error term in the expression of $\#M_{\underline{a},t}(B)$ (Theorem 3.1) is independent of $\underline{a}$ and t and that we can find an uniform upper bound of $C_{\underline{a},t}$ (also independent of $\underline{a}$ and t). This allows us to prove the convergence of (4) and afterwards to find a proper upper bound of $|\#M(B) - C \cdot B^{(n-1)/2}|$.

Acknowledgements

I would like to express my gratitude to my advisor professor Emmanuel Peyre for the many helpful conversations concerning this problem, and the Number Theory group at the University of Bristol (in particular professor Tim Browning and professor Trevor Wooley) for the useful tips concerning the circle method.

References

[1] D. Abramovich, Birational geometry for number theorists, in: Arithmetic Geometry, in: Clay Math. Proc., vol. 8, Amer. Math. Soc., Providence, RI, 2009, pp. 335–373.
[2] F. Campana, Fibres multiples sur les surfaces : aspects geométriques, hyperboliques et arithmétiques, Manuscripta Math. 117 (4) (2005) 429–461.
[3] H. Davenport, Analytic Methods for Diophantine Equations and Diophantine Inequalities, second edition, Cambridge Mathematical Library, Cambridge University Press, Cambridge, 2005. With a foreword by R.C. Vaughan, D.R. Heath-Brown and D.E. Freeman, edited and prepared for publication by T.D. Browning.
[4] B. Poonen, The projective line minus three fractional points, http://www-math.mit.edu/~poonen/slides/campana_s.pdf, July 2006.
[5] W.M. Schmidt, Analytische Methoden für Diophantische Gleichungen. Einführende Vorlesungen, DMV Seminar, vol. 5, Birkhäuser Verlag, Basel, 1984.