C. R. Acad. Sci. Paris, Ser. I 349 (2011) 611–614

Contents lists available at ScienceDirect

C. R. Acad. Sci. Paris, Ser. I

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Algebra/Group Theory

Odd character degrees for $\mathrm{Sp}(2n,2)$

Degrés de caractères impairs sur $\mathrm{Sp}_{2n}(2)$

Marc Cabanes

Institut de Mathématiques de Jussieu, Université Paris 7, 175 rue du Chevaleret, F-75013 Paris, France

ARTICLE INFO

Article history:
Received 22 April 2011
Accepted after revision 11 May 2011
Available online 12 June 2011

Presented by Jean-Pierre Serre

ABSTRACT

We check the McKay conjecture on character degrees for the case of symplectic groups over the field with two elements $\mathrm{Sp}_{2n}(2)$ and the prime 2. Then we check the inductive McKay condition (Isaacs–Malle–Navarro) for $\mathrm{Sp}_4(2^m)$ and all primes.

RÉSUMÉ

Nous vérifions la conjecture de McKay sur les degrés de caractères dans le cas des groupes symplectiques sur le corps à deux éléments $\mathrm{Sp}_{2n}(2)$ et du nombre premier 2. Nous montrons ensuite la condition de McKay inductive (Isaacs–Malle–Navarro) pour $\mathrm{Sp}_4(2^m)$ et tous les nombres premiers.

1. Introduction

If G is a finite group and ℓ is a prime number, denote by $\mathrm{Irr}_{\ell'}(G)$ the set of irreducible characters of G with degree prime to ℓ. The McKay conjecture asserts that

$$\left|\mathrm{Irr}_{\ell'}(G)\right| = \left|\mathrm{Irr}_{\ell'}\left(\mathrm{N}_G(P)\right)\right|$$

for P a Sylow ℓ-subgroup of G. This conjecture has gained new interest since appearance of Isaacs–Malle–Navarro's theorem reducing it to a related conjecture on quasi-simple groups (see [7]). The latter has been checked for all quasi-simple groups not of Lie type.

Among groups of Lie type and for ℓ being the defining prime, the group $\mathrm{Sp}_{2n}(2)$ had remained open (see [13]). This is the main purpose of this note (see Corollary 4 below). The method is by use of the Jordan decomposition of characters for the $\mathrm{Irr}_{\ell'}(G)$ side (see Proposition 2), while, for the $|\mathrm{Irr}_{\ell'}(\mathrm{N}_G(P))|$ side, we compute the abelian quotient of the Sylow 2-subgroup (Proposition 3), the latter an exception pointed by [6].

In a joint work with B. Späth, we developed some general methods which also cover the case $\mathrm{Sp}_{2n}(2^m)$ (see [2]) for $n > 2$, $m > 1$. Here, we present however the case of $\mathrm{Sp}_4(2^m)$ which requires some ad hoc analysis (see Section 3).

Notations. When ℓ is a prime and $n \geqslant 1$ an integer, one denotes by n_ℓ the greatest power of ℓ dividing n and $n_{\ell'} := n/n_\ell$. If H is a finite group and $X \subseteq \mathrm{Irr}(H)$, one denotes $X_{\ell'} := X \cap \mathrm{Irr}_{\ell'}(H)$.

If H acts on a set Y, one denotes by Y^H the subset of fixed points. For finite reductive groups $\mathbf{G}^F$ and their characters, we follow the notations of [4].

E-mail address: cabanes@math.jussieu.fr.

1631-073X/$ – see front matter
doi:10.1016/j.crma.2011.05.007

2. Odd character degrees for $\mathrm{Sp}_{2n}(2)$

Let us denote by $\mathbb{F}$ the algebraic closure of $\mathbb{F}_2$ the field with 2 elements. Let $n \geqslant 2$ be an integer, let $\mathbf{G} = \mathrm{Sp}_{2n}(\mathbb{F})$ with Frobenius endomorphism $F_0 : \mathbf{G} \to \mathbf{G}$ squaring matrix entries. Let $G = \mathbf{G}^{F_0} = \mathrm{Sp}_{2n}(\mathbb{F}_2)$, also denoted by $\mathrm{Sp}_{2n}(2)$ or $\mathrm{Sp}(2n, 2)$.

2.1. The global case

We refer to [4] for the notion of unipotent characters.

Let $n \geqslant 2$ be an integer. For our first lemma, see [9] 6.8.

Lemma 1. $\mathrm{Sp}_{2n}(2)$ *has five unipotent characters of odd degrees.*

Proposition 2. $\mathrm{Sp}_{2n}(2)$ *has* 2^{n+1} *characters of odd degrees.*

Proof. Recall $\mathbf{G} = \mathrm{Sp}_{2n}(\mathbb{F})$ with Frobenius endomorphism $F_0 : \mathbf{G} \to \mathbf{G}$ squaring matrix entries. Let $G = \mathbf{G}^{F_0} = \mathrm{Sp}_{2n}(2)$ (part of case (a) in [8] Section 8). Note that $\mathbf{G}$ has (trivial) connected center.

By [8] p. 164, $\mathbb{F}$ being of characteristic 2, there is an isogeny between $\mathbf{G}$ and its dual $\mathbf{G}^*$ inducing a bijection between rational semi-simple elements with isomorphism of centralizers of corresponding elements. This, along with property (A) of [8] 7.8 shows that $\mathrm{Irr}(G)$ is in bijection with the disjoint union of the $\mathcal{E}(\mathrm{C}_G(s), 1)$'s for s ranging over the semi-simple conjugacy classes of G (see [8] 8.7.6). Through this Jordan decomposition, the degrees are multiplied by $|\mathbf{G}^{*F_0}|_{2'} |\mathrm{C}_G(s)|_{2'}^{-1}$, so $|\mathrm{Irr}_{2'}(G)| = \sum_s |\mathcal{E}(\mathrm{C}_G(s), 1)_{2'}|$, a sum over the semi-simple classes of G.

Characteristic polynomials provide a bijection between the classes of semi-simple elements of $\mathrm{Sp}_{2n}(2)$ and the set of self dual polynomials $f \in \mathbb{F}_2[X]$ of degree $2n$. If s corresponds with f, then $\mathrm{C}_G(s) \cong \mathrm{Sp}_{2m}(2) \times C_s$ where C_s is a product of finite linear groups and $2m$ is the multiplicity of $(X-1)$ in f. For a given $m < n$, the number of such classes is 2^{n-m-1}. This is because one has to count the polynomials $f = (X-1)^{2m} g$ with a self dual $g(X) = 1 + a_1 X + \cdots + a_{n-m-1} X^{n-m-1} + a_{n-m} X^{n-m} + a_{n-m-1} X^{n-m+1} + \cdots + a_1 X^{2n-2m-1} + X^{2n-2m}$ such that $g(1) \neq 0$. Such g's are 2^{n-m-1}, corresponding to the choice of coefficients at degrees $1, 2, \ldots, n-m-1$ since $g(1) = a_{n-m}$ has to be $= 1$. For $m = n$ (central element) there is 1 conjugacy class ($s = 1$).

The unipotent characters of finite reductive groups of type A in characteristic 2 are of even degrees except the trivial character (see for instance [6] or [9] 6.8). Then Lemma 1 implies that each semi-simple class s corresponding with m as above satisfies $|\mathcal{E}(\mathrm{C}_G(s), 1)_{2'}| = 5$ for $m \geqslant 2$, $|\mathcal{E}(\mathrm{C}_G(s), 1)_{2'}| = 1$ otherwise. So the above indeed implies $|\mathrm{Irr}_{2'}(G)| = 5.\sum_{m=2}^{n-1} 2^{n-m-1} + 5 + 2^{n-2} + 2^{n-1} = 5.2^{n-2} + 3.2^{n-2} = 2^{n+1}$. $\square$

2.2. The local case

We use the description of $\mathrm{Sp}_{2n}(\mathbb{F}_2) \subset \mathrm{GL}_{2n}(\mathbb{F}_2)$ as the subgroup of matrices u such that ${}^t u \left(\begin{smallmatrix} 0 & J \\ J & 0 \end{smallmatrix}\right) u = \left(\begin{smallmatrix} 0 & J \\ J & 0 \end{smallmatrix}\right)$ where J denotes the matrix with coefficients $(\delta_{i,n+1-j})_{1 \leqslant i,j \leqslant n}$ and $u \mapsto {}^t u$ denotes transposition (see [4] 15.2). Let $U := \{\left(\begin{smallmatrix} x & xsJ \\ 0 & J\bar{x}J \end{smallmatrix}\right) \mid x \in V, s \in \mathrm{Sym}_n\}$ where Sym_n (resp. V) is the set of symmetric (resp. upper triangular unipotent) matrices of order n with coefficients in $\mathbb{F}_2$, and one denotes $\bar{x} = {}^t x^{-1}$. We have

Proposition 3. *U is a Sylow 2-subgroup of $G = \mathrm{Sp}_{2n}(2)$ for $n \geqslant 2$. Moreover $\mathrm{N}_G(U) = U$ and $U/[U, U]$ is of order 2^{n+1}.*

Corollary 4. *McKay conjecture (on character degrees) is satisfied in $G = \mathrm{Sp}_{2n}(2)$ for the prime 2 ($n \geqslant 2$). That is, the normalizer of any Sylow 2-subgroup of G has the same number of characters of odd degrees as G itself.*

Proof. By Proposition 3, the irreducible characters of $\mathrm{N}_G(U) = U$ of odd degrees are exactly the linear characters of U. So their number is the cardinality of $U/[U, U]$, that is 2^{n+1} thanks to Proposition 3 again. Combining with Proposition 2 gives our claim. $\square$

Proof of Proposition 3. Note that U equals the group of elements over $\mathbb{F}_2$ of a rational Borel subgroup (see [4] 15.2), so it equals its normalizer by the axioms of finite BN-pairs which are satisfied by this group. Thus our first claim.

Note also the semi-direct decomposition $U \cong \mathrm{Sym}_n \rtimes V$ for the action of V on Sym_n given by $x.s = xs^t x$ for $x \in V$, $s \in \mathrm{Sym}_n$. Since Sym_n is abelian and since the Sylow 2-subgroup V of $\mathrm{GL}_n(\mathbb{F}_2)$ is known to satisfy $|V/[V, V]| = 2^{n-1}$ (see for instance [4] p. 129 and [6]), our claim about $U/[U, U]$ reduces to show that $\mathrm{Sym}_n / [\mathrm{Sym}_n, V]$ is of order 4. So we have to prove that the sum $S' = \sum_{x \in V} \theta_x(\mathrm{Sym}_n)$ of images of endomorphisms $\theta_x : s \mapsto xs^t x - s$ of Sym_n has codimension 2.

For $1 \leqslant i, j \leqslant n$, let us denote by E_{ij} the usual elementary matrix of order n. We have $E_{ij} + E_{ji} + E_{ii} \in S'$ for any $1 \leqslant i < j \leqslant n$, by computing $\theta_x(s)$ for $s = E_{jj}$, $x = I_n + E_{ij}$. We also have $E_{ij} + E_{ji} \in S'$ for any $1 \leqslant i < j \leqslant n$ with $(i, j) \neq (n-1, n)$ (taking $s = E_{jk} + E_{kj}$ and $x = I_n + E_{ik}$ for some $k > i$, $k \neq j$). This shows that S' contains the $E_{ij} + E_{ji}$'s for $1 \leqslant i < j \leqslant n$ with $(i, j) \neq (n-1, n)$, along with $E_{11}, E_{22}, \ldots, E_{n-2,n-2}$ and $E_{n-1,n} + E_{n,n-1} + E_{n-1,n-1}$. This makes a subspace of codimension

2 in Sym_n, a supplement subspace being generated by $E_{n-1,n-1}$ and $E_{n,n}$. The action of V on the quotient is easily checked to be trivial (one just has to check the images of $E_{n-1,n-1}$ and $E_{n,n}$ by θ_x for $x = I_n + E_{ij}$ — which we just did above — since the latter generate V as a group, using again the fact that the field has two elements). So this subspace is indeed the sum of the images of all the θ_x's for $x \in V$. □

Theorem 5. *Let $n \geqslant 3$ be an integer. Then $\mathrm{Sp}_{2n}(2)$ is a simple group that satisfies the conditions of* [7] *Section* 10 *for all prime numbers.*

Proof. When $n = 3$, $\mathrm{Sp}_6(2)$ satisfies the theorem by [10] 4.1. When $n > 3$, $\mathrm{Sp}_{2n}(2)$ has trivial Schur multiplier and trivial outer automorphism group (see [5]), so that the checking required by [7] just amounts to the McKay conjecture itself (see [7] 10.3). For $\ell = 2$, it is Corollary 4. In the case of other primes, this is a consequence of Malle's parametrization [9] 7.8 along with Späth's extensibility results (see [11] 1.2, [12] 1.2, 8.4). □

3. $\mathrm{Sp}_4(2^m)$

Theorem 6. *Let $m \geqslant 2$ be an integer. Then $\mathrm{Sp}_4(2^m)$ is a simple group that satisfies the conditions of* [7] *Section* 10 *for all prime numbers.*

We keep $\mathbb{F}$ as above and let $\mathbf{G} = \mathrm{Sp}_4(\mathbb{F})$. We denote by $\mathbf{T}_0$ its diagonal torus and $(\mathbb{F}, +) \to \mathbf{G}$, $t \mapsto x_\alpha(t)$ its minimal unipotent $\mathbf{T}_0$-stable subgroups indexed by the $\mathbf{T}_0$-roots α. The Weyl group $W := \mathrm{N}_{\mathbf{G}}(\mathbf{T}_0)/\mathbf{T}_0$ is generated by the classes s_1, s_2 of the permutation matrices in $\mathrm{GL}_4(\mathbb{F})$ associated with the permutations $(1,2)$, $(3,4)$ and $(2,3)$, respectively.

Denote by F_0' the automorphism of $\mathbf{G} = \mathrm{Sp}_4(\mathbb{F})$ which sends $x_\alpha(t)$ to $x_{\alpha'}(t^2)$ if α is short (i.e. its associated reflection is conjugated with s_1), to $x_{\alpha'}(t)$ otherwise, and where $\alpha \mapsto \alpha'$ is the permutation of roots corresponding to the swap of s_1 and s_2, see [3] 12.3.3. Note that $F_0 = (F_0')^2$ (notation of Section 1). Denote $F = F_0^m$, so that $\mathrm{Sp}_4(2^m) = \mathrm{Sp}_4(\mathbb{F}_{2^m}) = \mathbf{G}^F$.

Proof of Theorem 6. The group $G = \mathrm{Sp}_4(2^m)$ $(m \geqslant 2)$ is simple with trivial Schur multiplier and cyclic outer automorphism group generated by F_0' (see [5]). Then the conditions of [7] Section 10 amount to find for each prime ℓ dividing $|G|$ a proper subgroup $N < G$ containing $\mathrm{N}_G(P)$ for P a Sylow ℓ-subgroup of G and such that $\sigma(N) = N$ and $|\mathrm{Irr}_{\ell'}(G)^\sigma| = |\mathrm{Irr}_{\ell'}(N)^\sigma|$ for any $\sigma \in \mathrm{N}_{\mathrm{Aut}(G)}(P)$ (see [1] Section 3). The case of $\ell = 2$ is also done in [1], so we assume that ℓ is odd dividing $(2^{4m} - 1)(2^{2m} - 1) = |\mathrm{Sp}_4(2^m)|_{2'}$. The order of 2^m mod ℓ is $e \in \{1, 2, 4\}$. Let $\mathbf{S}_e$ be a Sylow ϕ_e-torus of $\mathbf{G}$. We have that $\mathbf{T}_e := \mathrm{C}_{\mathbf{G}}(\mathbf{S}_e)$ is a maximal torus of $\mathbf{T}_0$-type $w_e = 1$, $s_1s_2s_1s_2$, or s_1s_2 according to e being 1, 2 or 4 (for types of maximal F-stable tori, and latter Levi subgroups, we refer to [4] p. 113).

Arguing as in the proof of [9] 5.14, any Sylow ℓ-subgroup P has a unique maximal toral elementary abelian subgroup whose normalizer N in G is then also $N := \mathrm{N}_G(\mathbf{S}_e) = \mathrm{N}_G(\mathbf{T}_e)$. It is stable by any automorphism σ such that $\sigma(P) = P$. From what has been said about possible σ's, and noting that N has an abelian normal subgroup $\mathbf{T}_e^F$ with ℓ' index, we see that we must just prove that

$$\left|\mathrm{Irr}_{\ell'}(G)^{F'}\right| = \left|\mathrm{Irr}(N)^{xF'}\right| \tag{E}$$

for any F' a power of F_0' and some $x \in G$ is such that $F'(\mathbf{S}_e) = \mathbf{S}_e^x$.

Bringing $(\mathbf{T}_e, F)$ to $(\mathbf{T}_0, w_eF)$ by conjugacy with some $g \in \mathbf{G}$ such that $g^{-1}F(g) \in w_e\mathbf{T}_0$, we may rewrite the above as

$$\left|\mathrm{Irr}_{\ell'}(G)^{F''}\right| = \left|\mathrm{Irr}\left(\mathrm{N}_{\mathbf{G}}(\mathbf{T}_0)^{w_eF}\right)^{F''}\right| \tag{E$'$}$$

when F'' is an isogeny commuting with w_eF and is in the same class as F' mod inner automorphisms of G.

Recall Malle's bijection $\mathrm{Irr}_{\ell'}(G) \xrightarrow{\sim} \mathrm{Irr}_{\ell'}(N)$ which, among other properties, sends components of $\mathrm{R}_{\mathbf{T}_e}^{\mathbf{G}}\theta$ to components of $\mathrm{Ind}_{\mathbf{T}_e}^N \theta$ for relevant $\theta \in \mathrm{Irr}(\mathbf{T}_e^F)$ (see [9] Section 7.1).

Let us first look at regular characters $\pm\mathrm{R}_{\mathbf{T}}^{\mathbf{G}}(\theta)$. They are of degree ℓ' if and only if $\mathbf{T}$ can be taken as $\mathbf{T}_e = \mathrm{C}_{\mathbf{G}}(\mathbf{S}_e)$ (see [9] 6.6). Such a character is fixed by F' if and only if $F'(\mathbf{T}_e, \theta)$ and $(\mathbf{T}_e, \theta)$ are $\mathbf{G}^F$-conjugate (see [1] Section 2.1.2). This is equivalent to $xF'(\theta)$ being $\mathrm{N}_G(\mathbf{S}_e)$-conjugate to θ ([9] 5.11). This is also the criterion for $\mathrm{Ind}_{\mathbf{T}^F}^N(\theta)$ being xF'-fixed as can be seen easily from the definition of induced characters. Thus our claim (E).

Let us now turn to unipotent characters. From [9] 6.5, we know that they have to be in $\mathcal{E}(\mathbf{G}^F, \mathbf{T}_e, 1)$, the set of irreducible characters occurring in the generalized character $\mathrm{R}_{\mathbf{T}_e}^{\mathbf{G}} 1$. So we have to check that $\mathcal{E}(\mathbf{G}^F, \mathbf{T}_e, 1)_{\ell'}^{F'}$ and $\mathrm{Irr}(N/\mathbf{T}_e^F)^{F'}$ have same cardinality.

As for the first set, one knows that among the six unipotent characters of $\mathrm{Sp}_4(2^m)$, only the two that are of generic degree $\frac{1}{2}q(q^2+1)$ are not fixed by F_0' (see [9] 3.9.a). Those are among unipotent characters of degree prime to ℓ only when $e = 1$ or 2. So it suffices to check that all characters of $N/\mathbf{T}_e^F$ but 2 are fixed by xF' in case $e = 1$ or 2 and F' is an odd power of F_0', and that all are fixed otherwise.

In cases $e = 1$ or 2, $w_1 = 1$, $w_2 = s_1s_2s_1s_2$ both are fixed by F_0', so one may take $F'' = F'$ in (E$'$) above. Recall that F_0' acts on W by permuting s_1 and s_2. The group W is dihedral of order 8, so F_0' induces an automorphism of order two of W^{ab}, so two linear characters out of four are F_0'-fixed, while the character of degree two is fixed. Hence our claim for any odd power of F'_0. In the case of an even power, the action is trivial, as expected. In the case $e = 4$, one may take $w_4 = s_1 s_2$ and $F'' = (s_1 F'_0)^a$ when $F' = (F'_0)^a$. Then the action of F'' on $(\mathrm{N}_{\mathbf{G}}(\mathbf{T}_0)/\mathbf{T}_0)^{w_4 F} = \mathrm{C}_W(w_4)$ is trivial.

We now assume $\mathcal{E}(G,s)^{F'}_{\ell'} \neq \emptyset$ for an s that is neither central nor regular. The group $\mathrm{C}_{\mathbf{G}}(s)$ is always a Levi subgroup of $\mathbf{G}$ (see proof of Proposition 2 above) and by [9] 6.5 it must contain a Sylow ϕ_e-torus. A proper F-stable Levi subgroup of $\mathbf{G}$ can contain a ϕ_1-Sylow for types $(\mathbf{L}_{\{s_1\}}, F)$ and $(\mathbf{L}_{\{s_2\}}, F)$ and a ϕ_2-Sylow for types $(\mathbf{L}_{\{s_1\}}, s_2 s_1 s_2 F)$ and $(\mathbf{L}_{\{s_2\}}, s_1 s_2 s_1 F)$. In each case the corresponding finite group has two unipotent characters, the trivial and the Steinberg characters, of distinct degrees, so that for an s whose class is F'-stable with such a centralizer in the dual, $\mathcal{E}(G,s)$ has two elements with distinct degrees, so F' acts trivially on $\mathcal{E}(G,s)$.

The corresponding statement on the local side is as follows: if θ is a non-regular non-central linear character of $\mathbf{T}_0^{w_e F}$, then $\mathrm{Ind}_{\mathbf{T}_0^{w_e F}}^{\mathrm{N}_{\mathbf{G}}(\mathbf{T}_0)^{w_e F}} \theta$ has two elements both F''-fixed if $F''(\theta) \in \mathrm{N}_{\mathbf{G}}(\mathbf{T}_0)^{w_e F}.\theta$. This holds because non-regularity implies $(\mathrm{N}_{\mathbf{G}}(\mathbf{T}_0)^{w_e F})_\theta / \mathbf{T}_0^{w_e F}$ is of order 2, but then F'' can act only trivially on it. □

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