C. R. Acad. Sci. Paris, Ser. I 349 (2011) 731–733

Contents lists available at ScienceDirect

C. R. Acad. Sci. Paris, Ser. I

www.sciencedirect.com

Algebra/Group Theory

A progenerator for representations of $\mathbf{SL}_n(\mathbb{F}_q)$ in transverse characteristic

Un progénérateur pour les représentations de $\mathbf{SL}_n(\mathbb{F}_q)$ en caractéristique transverse

Cédric Bonnafé

CNRS – UMR 5149, Institut de mathématiques et de modélisation de Montpellier, Université Montpellier 2, place Eugène-Bataillon, 34095 Montpellier cedex, France

ARTICLE INFO

Article history:
Received 27 March 2011
Accepted after revision 7 June 2011
Available online 13 July 2011

Presented by the Editorial Board

ABSTRACT

Let $G = \mathbf{GL}_n(\mathbb{F}_q)$, $\mathbf{SL}_n(\mathbb{F}_q)$ or $\mathbf{PGL}_n(\mathbb{F}_q)$, where q is a power of some prime number p, let U denote a Sylow p-subgroup of G and let R be a commutative ring in which p is invertible. Let $\mathrm{D}(U)$ denote the derived subgroup of U and let $e = \frac{1}{|\mathrm{D}(U)|}\sum_{u\in\mathrm{D}(U)} u$. The aim of this Note is to prove that the R-algebras RG and $eRGe$ are Morita equivalent (through the natural functor $RG\text{-mod} \longrightarrow eRGe\text{-mod}$, $M \mapsto eM$).

RÉSUMÉ

Soit $G = \mathbf{GL}_n(\mathbb{F}_q)$, $\mathbf{SL}_n(\mathbb{F}_q)$ ou $\mathbf{PGL}_n(\mathbb{F}_q)$, où q est une puissance d'un nombre premier p, soit U un p-sous-groupe de Sylow de G et soit R un anneau commutatif dans lequel p est inversible. Soit $\mathrm{D}(U)$ le groupe dérivé de U et soit $e = \frac{1}{|\mathrm{D}(U)|}\sum_{u\in\mathrm{D}(U)} u$. Le but de cette Note est de montrer que les R-algèbres RG et $eRGe$ sont Morita équivalentes (à travers le foncteur naturel $RG\text{-mod} \longrightarrow eRGe\text{-mod}$, $M \mapsto eM$).

Let n be a non-zero natural number, p a prime number, q a power of p and let $\mathbb{F}_q$ denote a finite field with q elements. Let $G_n = \mathbf{SL}_n(\mathbb{F}_q)$. We denote by U_n the group of $n \times n$ unipotent upper triangular matrices with coefficients in $\mathbb{F}_q$ (so that U_n is a Sylow p-subgroup of G_n). Let $\mathrm{D}(U_n)$ denote its derived subgroup: then, with $N = (n-1)(n-2)/2$,

$$\mathrm{D}(U_n) = \left\{ \begin{pmatrix} 1 & 0 & a_1 & \cdots & \cdots & a_{n-2} \\ 0 & 1 & 0 & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \ddots & a_N \\ \vdots & & & \ddots & 1 & 0 \\ 0 & \cdots & \cdots & \cdots & 0 & 1 \end{pmatrix} \;\middle|\; a_1, a_2, \dots, a_N \in \mathbb{F}_q \right\}.$$

We fix a commutative ring R in which p is invertible and we set

$$e_n = \frac{1}{|\mathrm{D}(U_n)|} \sum_{u\in\mathrm{D}(U_n)} u \in R\mathrm{D}(U_n).$$

E-mail address: cedric.bonnafe@math.univ-montp2.fr.
URL: http://ens.math.univ-montp2.fr/~bonnafe.

1631-073X/$ – see front matter
doi:10.1016/j.crma.2011.06.008

Then e_n is an idempotent of RG_n. The aim of this Note is to prove the following result (recall that an idempotent i of a ring A is called *full* if $A = AiA$):

Theorem 1. *If p is invertible in R, then e_n is a full idempotent of RG_n.*

Proof. First, let $R_0 = \mathbb{Z}[1/p]$, let ζ be a primitive p-th root of unity in $\mathbb{C}$ and let $\hat{R}_0 = R_0[\zeta]$. Let $\mathfrak{I}_0 = R_0G_ne_nR_0G_n$, $\hat{\mathfrak{I}}_0 = \hat{R}_0G_ne_n\hat{R}_0G_n$ and $\mathfrak{I} = RGeRG$. Since p is invertible in R, there is a unique morphism of rings $R_0 \to R$ which extends to a morphism of rings $R_0G_n \to RG_n$. So if $1 \in \mathfrak{I}_0$, then $1 \in \mathfrak{I}$. Also, as $(1, \zeta, \ldots, \zeta^{p-2})$ is an R_0-basis of $\hat{R}_0$, it is also an R_0G_n-basis of $\hat{R}_0G_n$. Therefore, if $1 \in \hat{R}_0G_ne_n\hat{R}_0G_n = \hat{R}_0 \otimes_{R_0} (R_0G_neR_0G_n)$, then $1 \in \mathfrak{I}_0$. Consequently, in order to prove Theorem 1, we may (and we shall) work under the following hypothesis:

Hypothesis. *From now on, and until the end of this proof, we assume that $R = \mathbb{Z}[1/p, \zeta]$.*

Now, let P_n denote the subgroup of $\mathbf{SL}_n(\mathbb{F}_q)$ defined by

$$P_n = \left\{ \left(\begin{array}{c|c} M & \begin{matrix} a_1 \\ \vdots \\ a_{n-1} \end{matrix} \\ \hline 0 \ \cdots \ 0 & 1 \end{array} \right) \;\middle|\; M \in \mathbf{SL}_{n-1}(\mathbb{F}_q) \quad \text{and} \quad a_1, \ldots, a_{n-1} \in \mathbb{F}_q \right\}.$$

Then $U_n \subset P_n$. We shall prove by induction on n that

$$e_n \textit{ is a full idempotent of } RP_n. \qquad (\mathcal{P}_n)$$

It is clear that Theorem 1 follows immediately from $(\mathcal{P}_n)$.

As $e_1 = 1$ and $e_2 = 1$, it follows that $(\mathcal{P}_1)$ and $(\mathcal{P}_2)$ hold. So assume that $n \geqslant 3$ and that $(\mathcal{P}_{n-1})$ holds. Let I_n denote the identity $n \times n$ matrix and let

$$V_n = \left\{ \left(\begin{array}{c|c} \mathrm{I}_{n-1} & \begin{matrix} a_1 \\ \vdots \\ a_{n-1} \end{matrix} \\ \hline 0 \ \cdots \ 0 & 1 \end{array} \right) \;\middle|\; a_1, \ldots, a_{n-1} \in \mathbb{F}_q \right\}.$$

Then $V_n \simeq (\mathbb{F}_q^+)^{n-1}$ and $P_n = \mathbf{SL}_{n-1}(\mathbb{F}_q) \ltimes V_n \simeq \mathbf{SL}_{n-1}(\mathbb{F}_q) \ltimes (\mathbb{F}_q^+)^{n-1}$. We set $V_n' = \mathrm{D}(U_n) \cap V_n$, so that $V_n' \simeq (\mathbb{F}_q^+)^{n-2}$ is normalized by P_{n-1}. Then

$$\mathrm{D}(U_n) = \mathrm{D}(U_{n-1}) \ltimes V_n'.$$

We now define

$$f_n = \frac{1}{|V_n'|} \sum_{v \in V_n'} v,$$

so that

$$e_n = e_{n-1} f_n.$$

By the induction hypothesis, there exist $g_1, h_1, \ldots, g_l, h_l$ in P_{n-1} and $r_1, \ldots, r_l$ in R such that

$$1 = \sum_{i=1}^{l} r_i g_i e_{n-1} h_i.$$

Therefore, as P_{n-1} normalizes V_n', it centralizes f_n and so

$$f_n = \left(\sum_{i=1}^{l} r_i g_i e_{n-1} h_i \right) f_n = \sum_{i=1}^{l} r_i g_i e_{n-1} f_n h_i = \sum_{i=1}^{l} r_i g_i e_n h_i.$$

So $f_n \in RP_ne_nRP_n$.

Let $\boldsymbol{\mu}_p$ denote the subgroup of $R^\times$ generated by ζ. If $\chi \in \mathrm{Hom}(V_n, \boldsymbol{\mu}_p)$, we define b_χ to be the associated primitive idempotent of RV_n:

$$b_\chi = \frac{1}{|V_n|} \sum_{v \in V_n} \chi(v)^{-1} v \in RV_n.$$

Then, as V_n is an elementary abelian p-group, we get

$$f_n = \sum_{\substack{\chi \in \mathrm{Hom}(V_n, \boldsymbol{\mu}_p) \\ \mathrm{Res}_{V_n'}^{V_n} \chi = 1}} b_\chi .$$

We fix a non-trivial element $\chi_0 \in \mathrm{Hom}(V_n, \boldsymbol{\mu}_p)$ whose restriction to V_n' is trivial. Then

$$b_{\chi_0} = b_{\chi_0} f_n \quad \text{and} \quad b_1 = b_1 f_n,$$

so b_1 and b_{χ_0} belong to $RP_n e_n RP_n$.

But $\mathbf{SL}_{n-1}(\mathbb{F}_q) \subset P_n$ has only two orbits for its action on $\mathrm{Hom}(V_n, \boldsymbol{\mu}_p)$ (because $n-1 \geqslant 2$): the orbit of 1 and the orbit of χ_0. Therefore, $b_\chi \in RP_n e_n RP_n$ for all $\chi \in \mathrm{Hom}(V_n, \boldsymbol{\mu}_p)$. Consequently,

$$1 = \sum_{\chi \in \mathrm{Hom}(V_n, \boldsymbol{\mu}_p)} b_\chi \in RP_n e_n RP_n,$$

as desired. $\square$

Finite reductive groups. Let $\mathbb{F}$ be an algebraic closure of $\mathbb{F}_q$, let $\mathbf{G}$ be a connected reductive group over $\mathbb{F}$ and let $F : \mathbf{G} \to \mathbf{G}$ be an isogeny such that some power F^δ is a Frobenius endomorphism relative to an $\mathbb{F}_q$-structure. We denote by $\mathbf{U}$ an F-stable maximal unipotent subgroup of $\mathbf{G}$ (it is the unipotent radical of an F-stable Borel subgroup). Define

$$e = \frac{1}{|\mathrm{D}(\mathbf{U})^F|} \sum_{u \in \mathrm{D}(\mathbf{U})^F} u \in R\mathbf{G}^F.$$

Theorem 2. *Assume that* $(\mathbf{G}, F)$ *is split of type* A. *Then* e *is a full idempotent of* $R\mathbf{G}^F$.

Proof. If $(\mathbf{G}, F)$ is split of type A, then we may assume that $\delta = 1$. Then there is a morphism of groups $\pi : G_n \to \mathbf{G}^F$ such that the image of U_n is $\mathbf{U}^F$. As $\mathrm{D}(\mathbf{U}^F) = \mathrm{D}(\mathbf{U})^F$ in this case, the extension of this morphism to the group algebras $\hat{\pi} : RG_n \to R\mathbf{G}^F$ sends e_n to e. By Theorem 1, the two-sided ideal of RG_n generated by e_n contains 1, so the result follows by applying $\hat{\pi}$. $\square$

Corollary 3. *If* $(\mathbf{G}, F)$ *is split of type* A, *then the functors*

$$\begin{array}{ccc} R\mathbf{G}^F\text{-mod} & \longrightarrow & eR\mathbf{G}^F e\text{-mod} \\ M & \longmapsto & eM \end{array} \quad \textit{and} \quad \begin{array}{ccc} R\mathbf{G}^F e\text{-mod} & \longrightarrow & R\mathbf{G}^F\text{-mod} \\ N & \longmapsto & R\mathbf{G}^F e \otimes_{eR\mathbf{G}^F e} N \end{array}$$

are mutually inverse equivalences of categories. In particular, $R\mathbf{G}^F$ *and* $eR\mathbf{G}^F e$ *are Morita equivalent, and* $R\mathbf{G}^F e$ *is a progenerator for* $R\mathbf{G}^F$.

Proof. This follows from Theorem 2 and, for instance, [3, Example 18.30]. $\square$

Example. Theorem 2 and Corollary 3 can be applied whenever $\mathbf{G}^F = \mathbf{GL}_n(\mathbb{F}_q)$, $\mathbf{SL}_n(\mathbb{F}_q)$ or $\mathbf{PGL}_n(\mathbb{F}_q)$.

Comments. (1) It is natural to ask whether Theorem 2 (or Corollary 3) can be generalized to other finite reductive groups. In fact, it cannot: indeed, if for instance $R = \mathbb{C}$, then saying that e is a full idempotent of $R\mathbf{G}^F$ means that every irreducible character of $\mathbf{G}^F$ is an irreducible component of an Harish-Chandra induced of some Gelfand–Graev character. But, if $\mathbf{G}$ is quasi-simple and $(\mathbf{G}, F)$ is not split of type A, then $\mathbf{G}^F$ admits a unipotent character which does not belong to the principal series: this character cannot be an irreducible component of an Harish-Chandra induced of a Gelfand–Graev character.

(2) In [1], a crucial step for the proof of a special case of the geometric version of Broué's abelian defect conjecture was [1, Theorem 4.1], where R. Rouquier and the author have proved the above Theorem 2 in the case where R is the integral closure of $\mathbb{Z}_\ell$ in a sufficiently large algebraic extension of $\mathbb{Q}_\ell$ (here, ℓ is a prime number different from p). The proof was essentially based on the classification, due to Dipper [2, 4.15 and 5.23], of simple modules for G_n in characteristic ℓ, and especially of cuspidal ones, which involves the Deligne–Lusztig theory.

The interest of the proof given here is that it does not rely on any classification of simple modules, and is based on elementary methods: as a by-product of this elementariness, Theorem 2 and Corollary 3 are valid over any commutative ring (in which p is invertible, which is a necessary condition if one wants the idempotent e_n to be well-defined).

References

[1] C. Bonnafé, R. Rouquier, Coxeter orbits and modular representations, Nagoya Math. J. 183 (2006) 1–34.
[2] R. Dipper, On quotients of Hom-functors and representations of finite general linear groups II, J. Algebra 209 (1998) 199–269.
[3] T.-Y. Lam, Lectures on Modules and Rings, Graduate Texts in Mathematics, vol. 189, Springer, 1999, xxiv+557 pp.