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# A Hamilton–Jacobi PDE in the space of measures and its associated compressible Euler equations

## *Une EDP de Hamilton–Jacobi dans l'espace des mesures et ses équations d'Euler compressibles associées*

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A B S T R A C T

We introduce a class of action integrals defined over probability-measure-valued path space. Minimal action exists in this context and gives weak solution to a compressible Euler equation. We prove that the Hamilton–Jacobi PDE associated with such variational formulation of Euler equation is well posed in viscosity solution sense.
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R É S U M É

Nous introduisons une classe d'intégrales d'action définies sur l'espace des chemins à valeurs mesures de probabilité. Dans ce contexte l'action minimale existe et donne une solution faible d'une équation d'Euler compressible. Nous montrons que l'équation de Hamilton Jacobi associ'ee à la formulation variationnelle de l'équation d'Euler est bien posée dans le sens des solutions de viscosité.
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## 1. Introduction

We denote by $\mathcal{P}_2(\mathbb{R}^d)$ the space of Borel probability measures over $\mathbb{R}^d$ with $\int_{\mathbb{R}^d} |x|^2 \rho(dx) < \infty$ endowed with the Wasserstein 2-metric $d$. $AC(0,T;\mathcal{P}_2(\mathbb{R}^d))$ is the class of $\mathcal{P}_2(\mathbb{R}^d)$-valued absolute continuous curves. Each $\rho(\cdot)$ in such class satisfies the continuity equation $\dot\rho := \partial_t \rho = -\operatorname{div}(\rho u)$ for some $u$ (Theorem 8.3.1 of [1]). This equation expresses a conservation of mass property and naturally introduces a class of parameterized curves, which motivates the following notion of tangent space and associated geometric structure on $\mathcal{P}_2(\mathbb{R}^d)$ (Chapter 8 of [1,6]):

$$H_{-1,\rho}(\mathbb{R}^d) := \{ m \in \mathcal{D}'(\mathbb{R}^d) \colon \|m\|_{-1,\rho} < \infty \}, \quad \|m\|^2_{-1,\rho} := \sup_{\varphi \in C_c^\infty(\mathbb{R}^d)} \{ 2\langle m, \varphi\rangle - \|\varphi\|^2_{1,\rho} \}. \tag{1}$$

In the above, $\|\varphi\|^2_{1,\rho} = \int_{\mathbb{R}^d} |\nabla\varphi|^2 \, d\rho$. It follows that $\int_{\mathbb{R}^d} |u|^2 \, d\rho = \|\dot\rho\|^2_{-1,\rho}$. We denote $\bar{\mathbb{R}} := \mathbb{R} \cup \{\pm\infty\}$.

**Definition 1.1** *(Gradient of a function).* Let $f : \mathcal{P}_2(\mathbb{R}^d) \mapsto \bar{\mathbb{R}}$, $\rho_0 \in \mathcal{P}_2(\mathbb{R}^d)$, and $f(\rho_0)$ be finite. We say that gradient of $f$ at $\rho_0$, denoted $\operatorname{grad} f(\rho_0)$, exists, if it can be identified as the unique element in $\mathcal{D}'(\mathbb{R}^d)$ satisfying the following property: for

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every $p \in C_c^\infty(\mathbb{R}^d)$ and the family of push forward of $\rho_0$ through the flow generated by $\nabla p$, i.e. $\{\rho^p(t) \in \mathcal{P}_2(\mathbb{R}^d):\ t \in \mathbb{R}\}$ with $\partial_t \rho^p + \operatorname{div}(\rho^p \nabla p) = 0$ and $\rho^p(0) = \rho_0$, we have $\lim_{t\to 0} t^{-1}(f(\rho^p(t)) - f(\rho^p(0))) =: \langle \operatorname{grad} f(\rho_0), p\rangle$.

Let $R(\rho\|\mu) := \int_{\mathbb{R}^d} \mathrm{d}\rho \log \frac{\mathrm{d}\rho}{\mathrm{d}\mu}$ denote relative entropy, define Gibbs measure $\mu^\Psi(\mathrm{d}x) := Z_\Psi^{-1} e^{-\Psi}$ with $Z_\Psi = \int_{\mathbb{R}^d} e^{-\Psi}\,\mathrm{d}x$, and entropy functional $S(\rho) := R(\rho\|\mu^\Psi)$. It follows then $\operatorname{grad} S(\rho) = -\Delta\rho - \operatorname{div}(\rho\nabla\Psi)$ whenever $S(\rho) < \infty$. Let $\psi := |\nabla\Psi|^2 - 2\Delta\Psi$, the Fisher information $I(\rho) := \|\operatorname{grad} S(\rho)\|_{-1,\rho}^2 = \int_{\mathbb{R}^d} \frac{|\nabla\rho|^2}{\rho}\,\mathrm{d}x + \int_{\mathbb{R}^d} \psi\,\mathrm{d}\rho$ (Appendix D.6 of [3]). Let $\nu > 0$, we introduce a modified kinetic energy $T(\rho,\dot\rho) := \frac{1}{2}\|\dot\rho + \nu \operatorname{grad} S(\rho)\|_{-1,\rho}^2$ to reinforce entropy dissipation (see [4] and its appendix). Let potential energy

$$V(\rho) := \int_{\mathbb{R}^d} \phi(x)\rho(\mathrm{d}x) + \frac{1}{2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \Phi(x-y)\rho(\mathrm{d}x)\rho(\mathrm{d}y) + \int_{\mathbb{R}^d} F\big(\rho(x)\big)\,\mathrm{d}x.$$

Without pursuing generality, we assume that $\Phi, \phi \in C^1(\mathbb{R}^d)$ have sub-quadratic growth, $\Phi(-x) = \Phi(x)$, $\Psi \in C^4(\mathbb{R}^d)$ is quasi-convex and that the leading order terms for both $\Psi$ and $\psi$ have polynomial growth of order bigger than 2 (e.g. $\Psi(x) = \frac{1}{4}|x|^4 - |x|^2$). Finally, let $F \in C^1$ be such that $|F(r)| \leqslant cr^\gamma$, $|rF'(r)| \leqslant c(1 + r^\gamma)$ for some finite $c \geqslant 0$ and some $\gamma \geqslant 1$ where $\gamma \in [1, 1 + \frac{2}{d})$ when $d \geqslant 3$ and $\gamma \in [1,2)$ when $d = 1,2$. For notational convenience, we set $V(\rho) = -\infty$ whenever $\rho \in \mathcal{P}_2(\mathbb{R}^d)$ has no Lebesgue density. The following is a consequence of Sobolev inequality and the fact that $\int_{\mathbb{R}^d}\rho(\mathrm{d}x) = 1$. See [4]:

**Lemma 1.2.** *There exists a right continuous nondecreasing sub-linear function $\zeta : R_+ \mapsto \mathbb{R}_+$ with $|V(\rho)| \leqslant \zeta(I(\rho))$. Moreover, $V$ is continuous on finite level sets of $I$.*

For $\rho(\cdot) \in AC(0,T;\mathcal{P}_2(\mathbb{R}^d))$ with $S(\rho(0)) < \infty$, by the calculus in [1], $\int_0^T T(\rho,\dot\rho)\,\mathrm{d}t = \frac{1}{2}\int_0^T (\|\dot\rho\|_{-1,\rho}^2 + \nu^2 I(\rho))\,\mathrm{d}t + \nu(S(\rho(T)) - S(\rho(0)))$. This observation motivates us considering Lagrangians $L$ and $\hat L : \mathcal{P}_2(\mathbb{R}^d) \times \mathcal{D}'(\mathbb{R}^d) \mapsto \mathbb{R}\cup\{+\infty\}$ by $L := T - V$ and $\hat L := \frac{1}{2}\|\dot\rho\|_{-1,\rho}^2 + \frac{\nu^2}{2} I - V$, where $\hat L$ is understood as $+\infty$ when $V = +\infty$. $\hat L$ takes value in $\mathbb{R}\cup\{+\infty\}$. $L$, however, is only well defined when $V$ is bounded from above in bounded sets of $\mathcal{P}_2(\mathbb{R}^d)$ (e.g. $F(r) \leqslant cr$ for some $c > 0$ will ensure this). Denote

$$A_T\big[\rho(\cdot)\big] := \int_0^T L(\rho,\dot\rho)\,\mathrm{d}t, \qquad J_T\big[\rho(\cdot)\big] := \int_0^T \hat L(\rho,\dot\rho)\,\mathrm{d}t, \quad \rho(\cdot) \in C\big([0,T];\mathcal{P}_2\big(\mathbb{R}^d\big)\big). \tag{2}$$

When both $A_T$ and $J_T$ are well defined and $S(\rho(0)) < \infty$, we have the following useful identity: $A_T[\rho(\cdot)] = J_T[\rho(\cdot)] + \nu(S(\rho(T)) - S(\rho(0)))$ for $\rho \in AC(0,T;\mathcal{P}_2(\mathbb{R}^d))$. Action minimizer for $A_T$ and $J_T$ are the same under mild conditions, and solves a compressible Euler equation (Theorem 2.1)

$$\begin{cases} \partial_t\rho + \operatorname{div}(\rho u) = 0 \\ \partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) + \nabla P(\rho) = -\rho\nabla(\phi + \Phi * \rho) - 2\nu^2\rho\nabla\left(\dfrac{\Delta\sqrt{\rho}}{\sqrt{\rho}} - \dfrac{1}{4}\psi\right) \\ P(\rho) = \rho F'(\rho) - F(\rho). \end{cases} \tag{3}$$

If $(\rho,u)$ are smooth for (3) to hold in classical sense, then it is also a weak solution as defined below.

**Definition 1.3** *(Weak solution).* $(\rho,u)$ is called a weak solution to system (3) if the following holds: $\rho(\cdot) \in AC(0,T;\mathcal{P}_2(\mathbb{R}^d))$ with $S(\rho(T)) + \int_0^T I(\rho(t))\,\mathrm{d}t < \infty$; $u : (0,T)\times\mathbb{R}^d \mapsto \mathbb{R}^d$ is Borel measurable satisfying $\int_0^T\int_{\mathbb{R}^d} |u(t,x)|^2\rho(t)\,\mathrm{d}t < \infty$; moreover, $\partial_t\rho + \operatorname{div}(\rho u) = 0$ holds in the distribution sense and

$$\int_0^T\int_{\mathbb{R}^d}\bigg[u(t,x)\cdot\big(\partial_t\xi(t,x) + (u\cdot\nabla)\xi(t,x)\big)\rho(t,x) + P(\rho)\operatorname{div}\xi - \big(\nabla(\phi + \Phi*\rho)\cdot\xi\big)\rho(t,x)$$
$$+ \nu^2\left(-\frac{\nabla\rho}{\rho}\cdot D\xi\cdot\frac{\nabla\rho}{\rho} + \Delta\operatorname{div}\xi + \frac{1}{2}\xi\cdot\nabla\psi\right)\rho(t,x)\bigg]\,\mathrm{d}x\,\mathrm{d}t = 0,$$

holds for every $\xi \in C_c^\infty((0,T)\times\mathbb{R}^d;\mathbb{R}^d)$, where $D\xi = (\partial_i\xi_j)_{(i,j)}$ is a matrix.

A satisfactory Hamilton–Jacobi PDE theory can also be developed (Theorem 2.2), based upon a Hamiltonian induced by the Lagrangian $L$, not the $\hat L$. For $V(\rho) < \infty$ and $n = -\operatorname{div}(\rho\nabla p)$ with $p \in C_c^\infty(\mathbb{R}^d)$, let

$$H(\rho,n) := \sup_{m\in H_{-1,\rho}(\mathbb{R}^d)} \big(\langle n,m\rangle_{-1,\rho} - L(\rho,m)\big) = -\big\langle \nu \operatorname{grad} S(\rho), n\big\rangle_{-1,\rho} + \frac{1}{2}\|n\|_{-1,\rho}^2 + V(\rho).$$

We do not attempt to extend $H$ to $(\rho, n) \in \mathcal{P}_2(\mathbb{R}^d) \times H_{-1,\rho}(\mathbb{R}^d)$, but rely on a delicate choice of test functions [3,2] to define the equations. Let $D_0 := \{f_0(\rho) = \frac{\theta}{2} d^2(\rho, \gamma) + \epsilon S(\rho) + c\colon\ c \in \mathbb{R}, \theta > 0, 0 < \epsilon < 2\nu, \gamma \in \mathcal{P}_2(\mathbb{R}^d)\}$ and $D_1 := \{f_1(\gamma) = -\frac{\theta}{2} d^2(\rho, \gamma) - \epsilon S(\gamma) + c\colon\ c \in \mathbb{R}, \theta > 0, 0 < \epsilon < 2\nu, \rho \in \mathcal{P}_2(\mathbb{R}^d)\}$. Denote $D := D_0 \cup D_1$. For each $f_0 \in D_0$ and $\rho$ in the effective domain of $f_0$ (i.e. $S(\rho) < \infty$), it can be proved that $\operatorname{grad} f_0(\rho) \in \mathcal{D}'(\mathbb{R}^d)$ exists. Furthermore, if $I(\rho) < \infty$, then $\operatorname{grad} f_0(\rho) \in H_{-1,\rho}(\mathbb{R}^d)$, and by Lemma 1.2, $H(\rho, \operatorname{grad} f_0(\rho))$ is finite. Similar relation also holds for $f_1 \in D_1$. Let $M(\mathcal{P}_2(\mathbb{R}^d); \overline{\mathbb{R}})$ denote the collection of measurable functions from $\mathcal{P}_2(\mathbb{R}^d)$ to $\overline{\mathbb{R}}$. We define operator $H : D \mapsto M(\mathcal{P}_2(\mathbb{R}^d); \overline{\mathbb{R}})$ as follows:

$$Hf(\rho) := \begin{cases} H\big(\rho, \operatorname{grad} f(\rho)\big) & \text{when } I(\rho) < \infty \\ -\infty & \text{when } I(\rho) = +\infty,\ f \in D_0 \\ +\infty & \text{when } I(\rho) = +\infty,\ f \in D_1. \end{cases} \tag{4}$$

**Lemma 1.4.** *$Hf_0 : \mathcal{P}_2(\mathbb{R}^d) \mapsto \mathbb{R} \cup \{-\infty\}$ is upper semicontinuous for $f_0 \in D_0$ and $Hf_1 : \mathcal{P}_2(\mathbb{R}^d) \mapsto \mathbb{R} \cup \{+\infty\}$ is lower semicontinuous for $f_1 \in D_1$.*

Let $\alpha > 0$ and for simplicity in statement of the results, we restrict attention to $h, g \in C_b(\mathcal{P}_2(\mathbb{R}^d))$. More general results can be found in [4]. By resolvent problem of the Hamilton–Jacobi PDE, we mean

$$f(\rho) - \alpha Hf(\rho) = h(\rho), \quad \rho \in \mathcal{P}_2\big(\mathbb{R}^d\big). \tag{5}$$

By Cauchy problem, we mean

$$\partial_t U(t, \rho) = HU(t, \rho), \quad (t, \rho) \in (0, T) \times \mathcal{P}_2\big(\mathbb{R}^d\big); \qquad U(0, \rho) = g(\rho) \quad \rho \in \mathcal{P}_2\big(\mathbb{R}^d\big). \tag{6}$$

**Definition 1.5** *(Resolvent problem).* Let $f \in M(\mathcal{P}_2(\mathbb{R}^d); \overline{\mathbb{R}})$; $|f| \leqslant \zeta(S)$ for some sub-linear function $\zeta : \mathbb{R}_+ \mapsto \mathbb{R}_+$; $f$ is continuous on finite level sets of $S$. Then

(i) $f$ is called a viscosity sub-solution to (5) if for each $f_0 \in D_0$ and $\rho_0 \in \mathcal{P}_2(\mathbb{R}^d)$ such that $(f - f_0)(\rho_0) = \sup_{\rho \in \mathcal{P}_2(\mathbb{R}^d)} (f - f_0)(\rho)$, we have $\alpha^{-1}(f - h)(\rho_0) \leqslant Hf_0(\rho_0)$.
(ii) $f$ is called a super-solution to (5) if for each $f_1 \in D_1$ and $\rho_1 \in \mathcal{P}_2(\mathbb{R}^d)$ such that $(f_1 - f)(\rho_1) = \sup_{\rho \in \mathcal{P}_2(\mathbb{R}^d)} (f_1 - f)(\rho)$, we have $\alpha^{-1}(f - h)(\rho_1) \geqslant Hf_1(\rho_1)$.

If $f$ is both sub- and super-solutions to (5), we call it a solution.

**Definition 1.6** *(Cauchy problem).* Let $U \in M([0, T] \times \mathcal{P}_2(\mathbb{R}^d); \overline{\mathbb{R}})$; $|U(t, \rho)| \leqslant \zeta(S(\rho))$ for all $(t, \rho) \in [0, T] \times \mathcal{P}_2(\mathbb{R}^d)$ for some sub-linear function $\zeta : \mathbb{R}_+ \mapsto \mathbb{R}_+$; $U$ is continuous on $[0, T] \times K_L$ where $K_L := \{\rho \in \mathcal{P}_2(\mathbb{R}^d)\colon\ S(\rho) \leqslant L\}$ for each $L < \infty$. Then

(i) $U$ is called a viscosity sub-solution to (6), if for each $U_0(t, \rho) = \frac{\alpha}{2}|t - s|^2 + f_0(\rho)$, and for each $(t_0, \rho_0) \in [0, T] \times \mathcal{P}_2(\mathbb{R}^d)$ such that $(U - U_0)(t_0, \rho_0) = \sup_{(t,\rho) \in [0,T] \times \mathcal{P}_2(\mathbb{R}^d)} (U - U_0)(t, \rho)$, we have
 (a) in the case of $t_0 > 0$, $(-\partial_t U_0 + HU_0)(t_0, \rho_0) \geqslant 0$;
 (b) in the case of $t_0 = 0$, $\limsup_{t \to 0+, \rho' \to \rho_0, S(\rho') \leqslant C} U(t, \rho') \leqslant g(\rho_0)$, for every $C \in \mathbb{R}_+$.
(ii) $U$ is called a super-solution to (6), if for each $U_1(s, \gamma) = -\frac{\alpha}{2}|t - s|^2 + f_1(\gamma)$ and for each $(s_0, \gamma_0) \in [0, T] \times \mathcal{P}_2(\mathbb{R}^d)$ such that $(U_1 - U)(s_0, \gamma_0) = \sup_{(s,\gamma) \in [0,T] \times \mathcal{P}_2(\mathbb{R}^d)} (U_1 - U)(s, \gamma)$, we have
 (a) in the case of $s_0 > 0$, $(-\partial_s U_1 + HU_1)(s_0, \gamma_0) \leqslant 0$;
 (b) in the case of $s_0 = 0$, $\liminf_{t \to 0+, \gamma' \to \gamma_0, S(\gamma') \leqslant C} U(t, \gamma') \geqslant g(\gamma_0)$, for every $C \in \mathbb{R}_+$.

If $U$ is both sub- and super-solutions, we call it a solution.

In view of growth estimate $|f| \leqslant \zeta(S)$, $\epsilon S(\rho) - f(\rho)$ is understood as $+\infty$, when $S(\rho) = +\infty$. Therefore, $f - f_0$ and $f_1 - f$ are always well defined on $\mathcal{P}_2(\mathbb{R}^d)$. The case of $U - U_0$ and $U_1 - U$ is handled similarly.

## 2. Main results

Let $P_t$ be the transition probability such that $\rho(t) := P_t \rho_0$ solves Fokker–Planck equation $\partial_t \rho = \Delta\rho + \operatorname{div}(\rho \nabla \Psi)$ with $\rho(0) = \rho_0$. We define $D(\rho_1 \| \rho_0; T) := \inf_{\pi \in \Pi(\rho_0, \rho_1)} R(\pi \| P_{\nu T} \otimes \rho_0)$ where $\Pi(\rho_0, \rho_1) \subset \mathcal{P}_2(\mathbb{R}^d \times \mathbb{R}^d)$ is the class of probability measures on $\mathbb{R}^d \times \mathbb{R}^d$ with first marginal $\rho_0$ and second marginal $\rho_1$. Ref. [4] proves the following:

**Theorem 2.1.** *Let $S(\rho_0) + D(\rho_1 \| \rho_0; T) < \infty$. Then there exists a $\rho(\cdot) \in AC(0, T; \mathcal{P}_2(\mathbb{R}^d))$ satisfying* $\inf\{J_T[\sigma(\cdot)]:\ \sigma(\cdot) \in C([0,T]; \mathcal{P}_2(\mathbb{R}^d)), \sigma(0) = \rho_0, \sigma(T) = \rho_1\} = J_T[\rho(\cdot)]$. *There exists a Borel vector field $u : (0,T) \times \mathbb{R}^d \mapsto \mathbb{R}^d$ with*

$$\int_0^T \int_{\mathbb{R}^d} \left| u(t,x) \right|^2 \mathrm{d}x \, \mathrm{d}t < \infty$$

*such that the pair $(\rho, u)$ is a weak solution (Definition 1.3) to (3). Additionally, if the extra condition $F(r) \leqslant cr$ holds for some $c > 0$, then* $\inf\{A_T[\sigma(\cdot)]:\ \sigma(\cdot) \in C([0,T]; \mathcal{P}_2(\mathbb{R}^d)), \sigma(0) = \rho_0, \sigma(T) = \rho_1\} = A_T[\rho(\cdot)] = J_T[\rho(\cdot)] + \nu(S(\rho_1) - S(\rho_0))$.

At least formally, one can show $u = -\nabla\varphi$ for some $\varphi$ [4]. Therefore, only potential flows are obtained.

**Theorem 2.2.** *There is at most one viscosity solution to (5) (respectively, to (6)) on $E_0 := \{\rho:\ S(\rho) < \infty\}$ (resp. $[0,T] \times E_0$). If $F(r) \leqslant cr$ for some $c > 0$, then $f(\rho_0) := \sup\{\int_0^\infty e^{-\alpha^{-1}s}(\alpha^{-1}h(\rho) - L(\rho, \dot{\rho}))\,\mathrm{d}s:\ \rho(\cdot) \in C([0,\infty); \mathcal{P}_2(\mathbb{R}^d)), \rho(0) = \rho_0\}$ (resp. $U(t, \rho_0) := \sup\{g(\rho(t)) - \int_0^t L(\rho(s), \dot{\rho}(s))\,\mathrm{d}s:\ \rho(0) = \rho_0,\ \rho(\cdot) \in C([0,\infty); \mathcal{P}_2(\mathbb{R}^d))\}$) is such a solution. Moreover, if $|\int_{\mathbb{R}^d} F(\rho(x))\,\mathrm{d}x| \leqslant \zeta(S(\rho))$ for some sub-linear $\zeta$, then the existence-uniqueness and continuity of solution above can be extended to $\mathcal{P}_2(\mathbb{R}^d)$.*

In the case of $V \equiv 0$, Theorem 2.2 also follows from results in [3,2]. A version of Theorem 2.1 also appears in mean-field game theory [5] using a different formulation (at individual particle level).

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