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Multifractal analysis of multiple ergodic averages

Analyse multifractale des moyennes ergodiques multiples

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ABSTRACT

In this Note we present a complete solution to the problem of multifractal analysis of multiple ergodic averages in the case of symbolic dynamics for functions of two variables depending on the first coordinate.

RÉSUMÉ

Dans cette Note nous présentons une solution complète au problème de l'analyse multifractale des moyennes ergodiques multiples dans le cas du système dynamique symbolique pour les fonctions de deux variables dépendant de la première coordonnée.

1. Introduction and results

Let $T : X \to X$ be a continuous map on a compact metric space X. Let $f_1, \dots, f_\ell$ ($\ell \geqslant 2$) be ℓ real valued continuous functions defined on X. We consider the following possible limits (for different $x \in X$):

$$M_{f_1,\dots,f_\ell}(x) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} f_1\big(T^k x\big) f_2\big(T^{2k} x\big) \cdots f_\ell\big(T^{\ell k} x\big). \tag{1}$$

Such limits are widely studied in ergodic theory. It was proposed in [2] to give a multifractal analysis of the multiple ergodic average $M_{f_1,\dots,f_\ell}(x)$. The authors of [2] succeeded in a very special case where $X = \{-1, 1\}^{\mathbb{N}}$, $f_k(x) = x_1$ for all k and T is the shift, by using Riesz products. In this Note, we shall study the shift map T on the symbolic space $X = \Sigma_m = S^{\mathbb{N}}$ with $S = \{0, 1, \dots, m-1\}$ ($m \geqslant 2$). We assume that $\ell = 2$ (the case $\ell \geqslant 3$ seems more difficult) and f_1 and f_2 are Hölder continuous. We endow Σ_m with the standard metric: $d(x, y) = m^{-n}$ where n is the largest $k \geqslant 0$ such that $x_1 = y_1, \dots, x_k = y_k$. The Hausdorff dimension of a set A in Σ_m will be denoted by $\dim A$.

For any $\alpha \in \mathbb{R}$, define

$$L(\alpha) = \big\{x \in \Sigma_m\colon\ M_{f_1,f_2}(x) = \alpha\big\}.$$

Let $\alpha_{\min} = \min_{x,y\in\Sigma_m} f_1(x) f_2(y)$ and $\alpha_{\max} = \max_{x,y\in\Sigma_m} f_1(x) f_2(y)$. Our question is to determine the Hausdorff dimension of $L(\alpha)$. We further assume that $\alpha_{\min} < \alpha_{\max}$ (otherwise both f_1 and f_2 are constant and the problem is trivial).

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From classical dynamical system point of view, the set $L(\alpha)$ is not standard and its dimension cannot be described by invariant measures supported on it. Let us first examine the largest dimension of ergodic measures supported on the set $L(\alpha)$ by introducing the so-called invariant spectrum:

$$F_{\mathrm{inv}}(\alpha) = \sup\bigl\{\dim\mu\colon\ \mu \text{ ergodic}, \mu\bigl(L(\alpha)\bigr) = 1\bigr\}.$$

Recall that (see [1])

$$\dim\mu = \inf\bigl\{\dim B\colon\ B \text{ Borel set}, \mu\bigl(B^c\bigr) = 0\bigr\}.$$

The dimension $F_{\mathrm{inv}}(\alpha)$ is in general smaller than $\dim L(\alpha)$ (compare the next two theorems). It is even possible that no ergodic measure is supported on $L(\alpha)$.

Theorem 1. *Let f_1 and f_2 be two Hölder continuous functions on Σ_m. If $L(\alpha)$ supports an ergodic measure, then*

$$F_{\mathrm{inv}}(\alpha) = \sup\left\{\dim\mu\colon\ \mu \textit{ ergodic}, \int f_1\,\mathrm{d}\mu \int f_2\,\mathrm{d}\mu = \alpha\right\}.$$

It is known [3] that the above supremum is the dimension of the set of points x such that

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n} f_1\bigl(T^k x\bigr)\cdot \lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n} f_2\bigl(T^k x\bigr) = \alpha.$$

Assume that f_1 and f_2 are the same function f. As a corollary of Theorem 1, $\mu(L(\alpha)) = 1$ for some ergodic measure μ implies $\alpha \geqslant 0$. So, if f takes a negative value $\alpha < 0$, then Theorem 1 shows that there is no ergodic measure supported on $L(\alpha)$. However, Theorem 2 shows that $\dim L(\alpha) > 0$.

In the following we assume that both f_1 and f_2 depend only on the first coordinate. For any $s \in \mathbb{R}$, consider the non-linear transfer equation

$$t_s(x)^2 = \sum_{Ty=x} e^{sf_1(x)f_2(y)} t_s(y). \tag{2}$$

It can be proved that the equation admits a unique solution $t_s : \Sigma_m \to \mathbb{R}_+$, which depends only on the first coordinate. Let $\mathrm{d}x$ denote the measure of maximal entropy for the shift on Σ_m and let

$$P(s) = \log\int_{\Sigma_m} t_s(x)\,\mathrm{d}x + \log m.$$

Also it can be proved that P is an analytic convex function and even strictly convex when $\alpha_{\min} < \alpha_{\max}$ (Lemma 3.1).

Theorem 2. *Let f_1 and f_2 be two functions on Σ_m depending only on the first coordinate. For $\alpha \notin [P'(-\infty), P'(+\infty)]$, we have $L(\alpha) = \emptyset$. For $\alpha \in [P'(-\infty), P'(+\infty)]$, we have*

$$\dim L(\alpha) = \frac{1}{2\log m}\bigl(P(s_\alpha) - s_\alpha P'(s_\alpha)\bigr)$$

where s_α is the unique solution of $P'(s) = \alpha$.

We can prove that $\alpha_{\min} \leqslant P'(-\infty) \leqslant P'(+\infty) \leqslant \alpha_{\max}$ and that $\alpha_{\min} = P'(-\infty)$ if and only if there exist $i_0, i_1, \dots, i_\ell \in S$ ($\ell \geqslant 1$) with $i_0 = i_\ell$ such that $f_1(i_k)f_2(i_{k+1}) = \alpha_{\min}$ (similar criterion for $\alpha_{\max} = P'(+\infty)$).

Let us look at two examples on Σ_2. For $f_1(x) = f_2(x) = 2x_1 - 1$, we have

$$\dim L(\alpha) = \frac{1}{2} + \frac{1}{2}H\left(\frac{1+\alpha}{2}\right), \qquad F_{\mathrm{inv}}(\alpha) = H\left(\frac{1+\sqrt{\alpha}}{2}\right)$$

where $H(x) = -x\log_2 x - (1-x)\log_2(1-x)$. See Fig. 1 for the graphs of $\dim L(\alpha)$ and $F_{\mathrm{inv}}(\alpha)$. Remark that $F_{\mathrm{inv}}(\alpha) = 0$ but $\dim L(\alpha) > 0$ for $-1 \leqslant \alpha < 0$. Also remark that $\dim L(\alpha)$ was computed in [2] by using Riesz products. See Fig. 2 for the graphs of $\dim L(\alpha)$ and $F_{\mathrm{inv}}(\alpha)$ when $f_1(x) = f_2(x) = x_1$. In the second case $F_{\mathrm{inv}}(\alpha) = H(\sqrt{\alpha})$ and $\dim L(\alpha)$ can be numerically computed through $P(s) = 2\log t_0(s)$ where $x = t_0(s)$ is the real solution of the third order algebraic equation

$$x^3 - 2x^2 - \bigl(e^s - 1\bigr)x + \bigl(e^s - 1\bigr) = 0.$$

These two examples show that $F_{\mathrm{inv}}(\alpha) < \dim L(\alpha)$ except for some special α's.

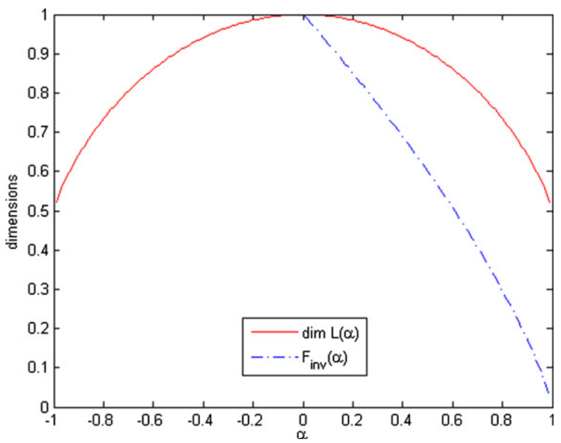

Fig. 1. When $f_1(x) = f_2(x) = 2x_1 - 1$.

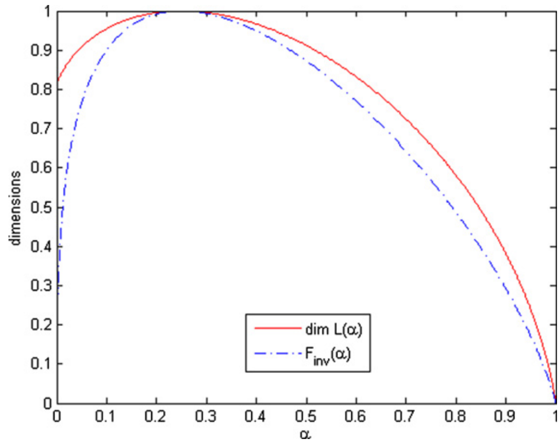

Fig. 2. When $f_1(x) = f_2(x) = x_1$.

The proof of Theorem 2 is based on the following observation. If f_1 and f_2 depend only on the first coordinate x_1, $\sum_k f_1(T^k x) f_2(T^{2k})$ can be decomposed into the sum of $\sum_j f_1(T^{i2^j} x) f_2(T^{i2^{j+1}} x)$ with odd i, which have independent coordinates. This observation was used in [2] to compute the box dimension of $X_0 = \{x\colon \forall n,\ x_n x_{2n} = 0\}$ which is a subset of $L(0)$ (here $f_1(x) = f_2(x) = x_1$ is considered). The Hausdorff dimension of X_0 was later computed in [5] where a non-linear transfer operator characterizes the measure of maximal Hausdorff dimension for X_0.

We have stated the results for functions of the form $f_1(x_1) f_2(y_1)$ (product of two functions depending on the first coordinate). But the results with obvious modifications hold for functions of the form $f(x_1, y_1)$.

2. Proof of Theorem 1

Let μ be an ergodic measure such that $\mu(L(\alpha)) = 1$. Then

$$\alpha = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \mathbb{E}_\mu\big[f_1\big(T^k x\big) f_2\big(T^{2k} x\big)\big] = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \mathbb{E}_\mu\big[f_1(x) f_2\big(T^k x\big)\big] = \mathbb{E}_\mu\big[f_1(x) M_{f_2}(x)\big].$$

(The first and third equalities are due to Lebesgue convergence theorem and the second one is due to the invariance of μ.) Since μ is ergodic, $M_{f_2}(x) = \mathbb{E}_\mu f_2$ for μ-a.e. x. So, $\alpha = \mathbb{E}_\mu f_1 \mathbb{E}_\mu f_2$. It follows that

$$F_{\mathrm{inv}}(\alpha) \leqslant \sup\{\dim \mu\colon\ \mu \text{ ergodic}, \mathbb{E}_\mu f_1 \cdot \mathbb{E}_\mu f_2 = \alpha\}.$$

To obtain the inverse inequality, it suffices to observe that the above supremum is attained by a Gibbs measure ν which is mixing and that the mixing property implies $M_{f_1, f_2}(x) = \mathbb{E}_\nu f_1 \cdot \mathbb{E}_\nu f_2$ ν-a.e.

3. Proof of Theorem 2

We will prove a result which is a bit more general than Theorem 2. Our proof is sketchy and a full proof is contained in [4] where other generalizations are also considered.

Here is the setting. Let $\varphi : S \times S \to \mathbb{R}$ be a non-constant function with minimal value $\alpha_{\min}$ and maximal value $\alpha_{\max}$. For $\alpha \in \mathbb{R}$, define

$$E(\alpha) = \left\{x \in \Sigma_m\colon\ \lim_{n\to\infty} n^{-1} \sum_{k=1}^{n} \varphi(x_k, x_{2k}) = \alpha\right\}.$$

Lemma 3.1. *For any $s \in \mathbb{R}$, the system*

$$t_i^2 = \sum_{j=0}^{m-1} e^{s\varphi(i,j)} t_j \quad (i = 0, 1, \ldots, m-1)$$

admits a unique solution $(t_0(s), t_1(s), \ldots, t_{m-1}(s))$ with strictly positive components, which is an analytic function of s. The function

$$P(s) = \log \sum_{j=0}^{m-1} t_j(s)$$

is strictly convex.

The proof of the lemma is lengthy. The existence and uniqueness of the solution are based on the fact that the square roots of right members of the system define an increasing operator on a suitable compact hypercube. The analyticity of the solution is a consequence of the implicit function theorem.

Theorem 3. *For any $\alpha \in [P'(-\infty), P'(+\infty)]$, we have*

$$\dim E(\alpha) = \frac{1}{2\log m}\big(P(s_\alpha) - s_\alpha P'(s_\alpha)\big)$$

where s_α is the unique solution of $P'(s) = \alpha$.

The solution $(t_0(s), t_1(s), \ldots, t_{m-1}(s))$ of the above system allows us to define a Markov measure μ_s with initial probability $(\pi(i))_{i\in S}$ and probability transition matrix $(p_{i,j})_{S\times S}$ defined by

$$\pi(i) = \frac{t_i(s)}{t_0(s) + t_i(s) + \cdots + t_{m-1}(s)}, \qquad p_{i,j} = e^{s\varphi(i,j)}\frac{t_j(s)}{t_i(s)^2}.$$

Now decompose the set of positive integers $\mathbb{N}^*$ into Λ_i (i being odd) with $\Lambda_i = \{i2^k\}_{k\geqslant 0}$ so that $\Sigma_m = \prod_{i:\ 2\nmid i} S^{\Lambda_i}$. Take a copy μ_s on each S^{Λ_i} and then define the product measure of these copies. This gives a probability measure $\mathbb{P}_s$ on Σ_m. Let $\underline{D}(\mathbb{P}_s, x)$ be the lower local dimension of $\mathbb{P}_s$ at x.

Lemma 3.2. *For any $x \in E(\alpha)$, we have $\underline{D}(\mathbb{P}_s, x) \leqslant \frac{1}{2\log m}[P(s) - \alpha s]$.*

It follows that $\dim E(\alpha) \leqslant \frac{1}{2\log m}[P(s) - \alpha s]$. Minimizing the right-hand side gives rise to

$$\dim E(\alpha) \leqslant \frac{1}{2\log m}\big[P(s_\alpha) - \alpha s_\alpha\big]$$

where s_α is the solution of $P'(s) = \alpha$. From the lemma, we can deduce that $L(\alpha) = \emptyset$ if $\alpha \notin [P'(-\infty), P'(+\infty)]$. In order to get the inverse inequality, we only have to show that $\mathbb{P}_{s_\alpha}$ is supported on $L(\alpha)$. We first prove the following law of large numbers by showing the exponential correlation decay of (F_n) under $\mathbb{P}_s$.

Lemma 3.3. *Let (F_n) be a sequence of functions defined on $S \times S$ such that $\sup_n \sup_{x,y} |F_n(x, y)| < \infty$. For $\mathbb{P}_s$-a.e. $x \in \Sigma_m$, we have*

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n}\big(F_k(x_k, x_{2k}) - \mathbb{E}_{\mathbb{P}_s} F_k(x_k, x_{2k})\big) = 0.$$

Applying the above lemma to $F_n(x_n, x_{2n}) = \varphi(x_n, x_{2n})$ for all n and computing $\mathbb{E}_{\mathbb{P}_s}\varphi(x_n, x_{2n})$, we get

Lemma 3.4. *For $\mathbb{P}_s$-a.e. $x \in \Sigma_m$, we have*

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n}\varphi(x_k, x_{2k}) = P'(s).$$

Thus we finished the proof for $\alpha \in (P'(-\infty), P'(+\infty))$. If $\alpha = P'(-\infty)$ (resp. $P'(+\infty)$), as in the standard multifractal analysis, we use the probabilities $\mathbb{P}_s$ and let s tend to $-\infty$ (resp. $+\infty$).

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