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Ordinary Differential Equations/Partial Differential Equations

On an explicit representation of the solution of linear stochastic partial differential equations with delays

Une expression analytique pour la solution d'équations aux dérivées partielles linéaires stochastiques avec délais

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ABSTRACT

Based on the analysis of a certain class of linear operators on a Banach space, we provide a closed form expression for the solutions of certain linear partial differential equations with non-autonomous input, time delays and stochastic terms, which takes the form of an infinite series expansion.

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R É S U M É

En se basant sur l'analyse d'une certaine classe d'opérateurs linéaires dans des espaces de Banach, nous établissons une expression analytique pour la solution de certaines équations aux dérivées partielles linéaires avec des entrées non-autonomes, des délais et des termes stochastiques, sous la forme d'un développement en série.

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Les systèmes différentiels linéaires sont particulièrement importants en mathématiques ainsi que dans leurs applications, soit en tant que modèles, approximations, ou bien car ils gouvernent la stabilité de solutions de systèmes non-linéaires. Outre les cas les plus simples d'équations différentielles ordinaires, il est difficile d'accéder à la solution de tels systèmes. Dans ce manuscrit nous présentons une formulation explicite des solutions d'équations différentielles linéaires, stochastiques, à délais, non-autonomes, en dimension quelconque. Ce développement est particulièrement utile pour l'analyse de la dynamique de réseaux de neurones linéaires, bruités et faiblement connectés, problème qui a motivé ce travail plus théorique.

Ce papier se place à l'intersection de l'analyse fonctionnelle et de l'algèbre linéaire. Dans la Section 2 nous introduisons dans un premier temps les notations et le cadre de travail fonctionnel : nous considérons l'action d'opérateurs linéaires sur des fonctions spatio-temporelles. La Proposition 2.1 fournit un outil technique pour inverser une certaine classe d'opérateurs linéaires dont les parties spatiales et temporelles peuvent être séparées. La mécanique de la preuve est basée sur la notion de produit de Kronecker généralisée aux fonctions continues en temps.

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Dans la Partie 3 nous montrons que cette classe d'opérateurs inversibles est assez large pour contenir le cas des équations différentielles, linéaires, stochastiques, à délais. En effet, on montre que la transformée de Fourier de ces equations nous ramène aux hypothèses de la Proposition 2.1. La preuve est principalement basée sur la co-diagonalisabilité des opérateurs de convolution et des opérateurs différentiels dans la base de Fourier. Elle mène à la formule explicite (4) valable sous une certaine condition spectrale (1) que doit vérifier l'opérateur spatial (voir aussi la Fig. 1 qui présente quelques exemples).

Cette formule analytique, qui se présente sous la forme d'une série convergente, permet de calculer des approximations de la solution réelle en tronquant simplement les termes d'ordre élevé. L'approximation est d'autant plus précise que l'opérateur spatial est proche d'une homothétie. Cette expression permet aussi de développer une nouvelle classe de méthodes numériques pour résoudre de telles équations, ne faisant intervenir que des sommes et produits de matrices (donc facilement parallélisables par opposition aux méthodes traditionnelles). Enfin on illustre ce résultat général par quelques exemples concrets d'applications dans les Sections 3.3–3.4.

1. Introduction

Linear differential systems are ubiquitous in pure and applied mathematics, either as models, approximations, but also because the stability of solutions of non-linear differential systems reduces to the study of linear systems. Such systems might include stochastic terms (see [4]), temporal delays (see [3]), and also encompass the case of partial differential equations. Apart from the simplest linear finite-dimensional differential equations, finding closed forms expressions for the solutions of general linear differential systems is very complex. In this paper, based on the treatment of evolution equations as algebraic equations in a suitable Banach space, we propose a closed form expression for the solution of linear, non-autonomous, stochastic, time-delayed partial differential systems. Application of this framework to several classical examples such as the delayed Ornstein–Uhlenbeck process or the stochastic heat equation are developed in Sections 3.3 and 3.4. This expression is especially useful to understand the dynamics of weakly connected linear learning neural networks, problem which motivated the development of this more general framework and allowed to uncover the structure of equilibrium connectivities evolving under Hebbian learning (see [2]).

2. Framework and general result

The framework we develop here is based on extending notions of matrix calculus to infinite-dimensional spaces. The linearity of the equation motivates to extend some finite-dimensional linear algebra and matrix concepts to infinite-dimensional spaces.

We consider in the manuscript linear equations in a Banach space \mathcal{C} of real functions of time t and a variable $x \in E$, called *space variable*, where E can either be a finite set $\{1, \dots, N\}$ (in which case \mathcal{C} is equivalent to the space of \mathbb{R}^N -valued functions), countable or continuous, typically \mathbb{R} , in which case \mathcal{C} is a space of two-variable functions. The particular problem under consideration governs the choice of the space \mathcal{C} , in particular including regularity or integrability properties (typically \mathcal{C} is an L^p or a Sobolev space). Similarly to a matrix notation, we denote the value of $X \in \mathcal{C}$ at $(x, t) \in E \times \mathbb{R}$ by X_{xt} .

Let \mathcal{E} denote the space of bounded linear operators on \mathcal{C} . We are interested in solving equations of type $\mathcal{L}X = B$ where $\mathcal{L} \in \mathcal{E}$ (this operator may involve differentials in time and/or space) and $B \in \mathcal{C}$. We will restrict the study to a class of operators of a particular form we now detail. To this end, we introduce two kinds of linear operators on \mathcal{C} : the *space operators* L acting on the first (space) variable, i.e. linear operators on \mathbb{R}^E . If E is finite, this set is reduced to the matrices. If E is equal to \mathbb{R}^d , it contains all the linear operators acting on functions of the space variable, in particular, under suitable regularity conditions, integral or differential operators. The action of the space operators L on a function $X \in \mathcal{C}$ is denoted $L \cdot X$ (acting on the left). The *time operators* essentially act on the second (time) variable, and the transform might depend on the space variable x . In other words, these transforms \mathcal{R} can be represented by a family of operators $(\mathcal{R}_x, x \in E)$ such that for any x , \mathcal{R}_x is a linear operator on $L^2(\mathbb{R})$. The action of a time operator \mathcal{R} on $X \in \mathcal{C}$ is written $X \cdot \mathcal{R}$ (acting on the right). In the paper, we will mainly be interested in diagonalizable time operators. Diagonal operators in the time domain are operators \mathcal{R} whose action can be written in the form $(X \cdot \mathcal{R})_{xt} = r(x, t)X_{xt}$. This class includes for instance all linear differential time operators, which are diagonalizable in the Fourier basis. Another class of time operators we will be considering is the class \mathcal{CO} of convolution operators with respect to time. Given a finite measure g of \mathbb{R} , the convolution operator $\mathcal{T}_g \in \mathcal{CO}$ associated with g is defined as $(X \cdot \mathcal{T}_g)_{xt} = \int_{-\infty}^{\infty} X_{x(t-s)} dg(s)$. Such operators are generalizations of Toeplitz matrices generated by g , with, loosely speaking, infinitely many rows and columns. An important property of the convolution operators is that they are diagonal in the Fourier basis.

For L a space operator and \mathcal{R} a time operator, we define the Kronecker product $L \otimes \mathcal{R}$ as the mixed operator of \mathcal{E} such that $(L \otimes \mathcal{R})(X) = L \cdot (X \cdot \mathcal{R})$. Note that the product becomes associative when \mathcal{R} is a convolution operator which will be the case in Section 3. This definition extends the property of vectorization of the Kronecker product of matrices in linear algebra (see e.g. [1]).

The main technical result of the paper is given in the following:

Proposition 2.1. *Let $\mathcal{L} = A \otimes B + Id_{\mathcal{C}} \otimes \mathcal{D}$ be a linear operator, for A a space operator and B, \mathcal{D} co-diagonalizable time operators, with B invertible. For the sake of simplicity, we assume that they are diagonal in the natural time basis, and denote for $x \in E$, $B_x = \text{diag}_{t \in \mathbb{R}}(b(x, t))$ and $\mathcal{D}_x = \text{diag}_{t \in \mathbb{R}}(d(x, t))$. We assume that $\inf_{x,t} |b(x, t)| > 0$ and the spectral condition:*

$$\exists l \in \mathbb{R}^* \text{ such that } \lambda \stackrel{\text{def}}{=} \frac{\|W\|}{\inf_{x,t} \left\{ \left| l - \frac{d(x,t)}{b(x,t)} \right| \right\}} < 1 \tag{1}$$

where $W \stackrel{\text{def}}{=} lId_{\mathbb{R}^E} + A$ and $\|W\| = \sup_{X \neq 0} \frac{\|W \cdot X\|}{\|X\|}$ is the operator norm. Then $A \otimes B + Id_C \otimes D$ is invertible and its inverse reads:

$$(A \otimes B + Id_C \otimes D)^{-1} = - \sum_{k=0}^{+\infty} W^k \otimes \text{diag}_{t \in \mathbb{R}} \left(\frac{1}{b(x,t) \left(l - \frac{d(x,t)}{b(x,t)} \right)^{k+1}} \right) \tag{2}$$

Remark. The spectral condition is merely a technical sufficient condition for the convergence of the series. The optimal choice of the parameter l minimizes λ . This choice will be discussed in different cases in Section 3.3. The relatively formal setting and assumptions will become clearer in the applications, Section 3.

Proof. The direct introduction of the inverse can appear artificial at first sight. However, this formula is a natural extension of the discrete-time case where direct linear algebra and Kronecker products calculations quite simply provide a closely related expression the interested reader can readily derive.

In order to prove the proposition, we first need to prove that the operator indeed exists, and that it constitutes the inverse of \mathcal{L} . It is easy to show that under the assumption of the proposition that the sequence of operators in \mathcal{E} defined by: $M_N \stackrel{\text{def}}{=} - \sum_{k=0}^N W^k \otimes \text{diag}_{t \in \mathbb{R}} \left(\frac{1}{b(x,t) \left(l - \frac{d(x,t)}{b(x,t)} \right)^{k+1}} \right)$ constitutes a Cauchy sequence in \mathcal{E} . Since \mathcal{C} is a Banach space, so is \mathcal{E} , and hence the sequence $(M_n)_n$ converges. The limit of this sequence is our inverse candidate, and is denoted as the infinite series (2).

In order to prove that this limit is indeed the inverse of \mathcal{L} , we compute the limit of $(M_N \circ \mathcal{L})X$ (or similarly $(\mathcal{L} \circ M_N)X$) for a given $X \in \mathcal{C}$. It is easy to show, developing the series, that we have:

$$((M_N \circ \mathcal{L})X)_{xt} = - \sum_{k=0}^N -W^k \cdot \frac{X_{.,t}}{\left(l - \frac{d(.,t)}{b(.,t)} \right)^k} + W^{k+1} \cdot \frac{X_{.,t}}{\left(l - \frac{d(.,t)}{b(.,t)} \right)^{k+1}} = X_{xt} - W^{N+1} \cdot \frac{X_{.,t}}{\left(l - \frac{d(.,t)}{b(.,t)} \right)^{N+1}}$$

where $Y_{.,t}$ for $Y \in \mathcal{C}$ denotes the application $E \mapsto \mathbb{R}$ such that $Y_{.,t}(x) = Y_{xt}$. Here again, the assumptions of the proposition ensure that the second term vanishes as N goes to infinity. \square

3. Application to solving linear time-delayed stochastic partial differential equations

In this section we make explicit the use of the inversion formula (2) in the case of linear delayed, stochastic, partial differential equations. Several examples with different convolution operators will illustrate the main result of the section stated in Theorem 3.1.

3.1. General result

Let \mathcal{X} be a Hilbert space, typically \mathbb{R}^n for $n \in \mathbb{N}$, $L^2(\mathbb{R}^n)$ or a Sobolev space of applications on \mathbb{R}^n . We consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ satisfying the usual conditions and B a standard adapted \mathcal{X} -Brownian motion (for the existence and properties of this object in infinite-dimensional spaces, see [5, Chapter 4]). We aim at solving the non-autonomous time-delayed stochastic differential equation:

$$\begin{cases} dX = (A \cdot (X * g) + I) dt + \Sigma \cdot dB \\ X|_{\mathbb{R}_-} = \zeta_0 \in L^2_{\mathcal{X}}(\mathbb{R}^*_-) \end{cases} \tag{3}$$

with $\Sigma : \mathcal{X} \mapsto \mathcal{X}$ linear, $I \in C(\mathbb{R}_+, \mathcal{X})$ an external input, g a finite measure of the real line supported on \mathbb{R}_+ , i.e. a causal measure, and $*$ denoting the convolution. Existence and uniqueness of weak solutions for such equations is ensured, see e.g. [4,5]. We consider the case where the system has a unique strong solution. In the case where $\mathcal{X} = \mathbb{R}^n$, this occurs under the assumptions of the section, see e.g. [4, Chapter 5], and in the infinite-dimensional case, we need to assume that B is a genuine Wiener process (i.e. the trace of the covariance matrix is finite, and the initial condition is in the domain of A , see [5]). The solution of this stochastic differential equation at time $t \in \mathbb{R}_+$ is defined by the integral equation $X(t) = \zeta_0(0) + \int_0^t (A \cdot (X * g)(s) + I(s)) ds + \int_0^t \Sigma \cdot dB$ and $X|_{\mathbb{R}_-} = \zeta_0$.

This problem can be set in the framework described in Section 2 using a transformation inspired by the classical Fourier transform of the solution in the time domain. To perform this transformation rigorously in our particular stochastic setting, we stop our processes at a finite time $\tau > 0$. We define $X_\tau : t \in \mathbb{R} \rightarrow \mathbb{1}_{[0,\tau]}(t)X(t)$ the restriction of X to the compact support $[0, \tau]$ and null elsewhere. Similarly, define $I_\tau = \mathbb{1}_{[0,\tau]}I$ and $dB_\tau = \mathbb{1}_{[0,\tau]}dB$. We have:

Theorem 3.1. For all $\tau \in \mathbb{R}_+$, choose $l \in \mathbb{R}^*$ and W a space operator such that $W = lId_C + A$. If the spectral condition (1) is satisfied, i.e. in the present case $\|W\| < \inf_{\xi} \left\{ \left| l + \frac{2i\pi\xi}{g(\xi)} \right| \right\}$, then the solution of Eq. (3) is given by

$$X_\tau = \sum_{k=0}^{+\infty} W^k \cdot (\zeta_0(0)\delta_0 + \tilde{I}_\tau + \Sigma \cdot dB_\tau) \cdot \mathcal{U} \cdot \mathcal{V}^k \tag{4}$$

where $\mathcal{U} = \mathcal{F} \cdot \text{diag}_{\xi \in \mathbb{R}} \left(\frac{1}{l\hat{g}(\xi) + 2i\pi\xi} \right) \cdot \mathcal{F}^{-1}$, $\mathcal{V} = \mathcal{F} \cdot \text{diag}_{\xi \in \mathbb{R}} \left(\frac{\hat{g}(\xi)}{l\hat{g}(\xi) + 2i\pi\xi} \right) \cdot \mathcal{F}^{-1}$ and $\tilde{I}_\tau = I_\tau + A \cdot (\zeta_0 * g)$. The notation $(dB_\tau \cdot \mathcal{U})_{xt}$ stands for the square integrable stochastic integral $\int_0^{\min(t, \tau)} \mathcal{U}_{st} dB(s)$ on $[0, \tau[$.

Remark. The convergence of the series (4) occurs as soon as the spectral condition (1) is satisfied on the subspace spanned by $W^k \cdot (\zeta_0(0)\delta_0 + \tilde{I}_\tau + \Sigma \cdot dB_\tau) \cdot \mathcal{U} \cdot \mathcal{V}^k$.

Proof. First, note that $A \cdot (X * g) = A \cdot (X_\tau * g) + A \cdot (\zeta_0 * g)$ yielding the equation on X_τ : $dX_\tau = (A \cdot (X_\tau * g) + \tilde{I}_\tau) dt + \Sigma \cdot dB_\tau$. Thus, the initial condition on X acts as an external input on X_τ . In the deterministic finite-dimensional case, it is well known that differential operators are diagonal in the Fourier basis. Based on this result, we introduce the Fourier transform \mathcal{F} of Eq. (3) for a fixed $\omega \in \Omega$. As mentioned, for almost all $\omega \in \Omega$, the processes involved are bounded, hence the function of time, on the compact interval $[0, \tau]$, is square integrable in time. Let $Z^\xi : t \in \mathbb{R} \rightarrow e^{-2i\pi t\xi} X_\tau(t)$ for $\xi \in \mathbb{R}$ the Fourier variable and X is the unique solution of Eq. (3). Itô formula yields for $t < \tau$

$$dZ^\xi(t) = (-2i\pi\xi Z^\xi(t) + A \cdot (Z^\xi * g)(t) + e^{-2i\pi t\xi} \tilde{I}_\tau(t)) dt + e^{-2i\pi t\xi} \Sigma \cdot dB_\tau(t) \tag{5}$$

Let us denote by $\hat{X}_\tau : \xi \in \mathbb{R} \rightarrow \int_0^\tau Z^\xi(s) ds$ the Fourier transform of X_τ and $\hat{I}_\tau : \xi \rightarrow \int_0^\tau e^{-2i\pi t\xi} \tilde{I}_\tau(t) dt$. The process \hat{B}_τ is the well-defined stochastic integral $\int_0^\tau e^{-2i\pi t\xi} dB(t)$. The integral form of Eq. (5), using the fact that the convolution is diagonal in the Fourier basis, denoting $\hat{D} = \text{diag}_{\xi \in \mathbb{R}}(-2i\pi\xi)$ and $\hat{G} = \text{diag}_{\xi \in \mathbb{R}}(\hat{g}(\xi))$, leads to the functional equation:

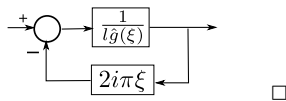
$$Z(\tau) - Z(0) = A \cdot \hat{X}_\tau \cdot \hat{G} + \hat{X}_\tau \cdot \hat{D} + \hat{I}_\tau + \Sigma \cdot \hat{B}_\tau$$

Applying Proposition 2.1 for a fixed $\omega \in \Omega$ where \mathcal{C} is the set of square integrable functions on $[0, \tau[$ which is a Banach space, we obtain:

$$\hat{X}_\tau = \sum_{k=0}^{+\infty} W^k \cdot (-Z(\tau) + Z(0) + \hat{I}_\tau + \Sigma \cdot \hat{B}_\tau) \cdot \text{diag}_{\xi \in \mathbb{R}} \left(\frac{1}{\hat{g}(\xi) \left(l + \frac{2i\pi\xi}{\hat{g}(\xi)} \right)^{k+1}} \right) \tag{6}$$

We now take the inverse Fourier transform of this expression by applying the time operator \mathcal{F}^{-1} . First of all, we perform the inversion on the terms $\hat{I}_\tau = \tilde{I}_\tau \cdot \mathcal{F}$. It is easy to show that $\hat{I}_\tau \cdot \text{diag} \left(\frac{1}{\hat{g}(\xi) \left(l + \frac{2i\pi\xi}{\hat{g}(\xi)} \right)^{k+1}} \right) \cdot \mathcal{F}^{-1} = \tilde{I}_\tau \cdot \mathcal{U} \cdot \mathcal{V}^k$. Similarly, the term $(\hat{B}_\tau \cdot \text{diag} \left(\frac{1}{\hat{g}(\xi) \left(l + \frac{2i\pi\xi}{\hat{g}(\xi)} \right)^{k+1}} \right) \cdot \mathcal{F}^{-1})_t$ can be written $dB_\tau \cdot \mathcal{U} \cdot \mathcal{V}^k$.

Moreover, for $x \in \{0, \tau\}$ an easy computation shows that $(Z(x) \cdot \text{diag} \left(\frac{1}{\hat{g}(\xi) \left(l + \frac{2i\pi\xi}{\hat{g}(\xi)} \right)^{k+1}} \right) \cdot \mathcal{F}^{-1})_t = (X(x)\delta_x) \cdot \mathcal{U} \cdot \mathcal{V}^k$. Furthermore, the operators \mathcal{U} and \mathcal{V} are causal, i.e. if Y has a support $\subset [c, +\infty[$ then $Y \cdot \mathcal{U} \cdot \mathcal{V}^k$ also has a support $\subset [c, +\infty[$. Indeed, $\hat{u} : \xi \mapsto \frac{1}{l\hat{g}(\xi) + 2i\pi\xi}$ corresponds to the transfer function of a closed loop filter shown on the right, and hence \mathcal{U} is clearly causal since g is. \mathcal{V} is also causal as the convolution of g and \mathcal{U} . This implies that the contribution of $Z(\tau)$ vanishes in Eq. (6) since it has its support in $[\tau, \infty[$.



3.2. Computational remarks

Truncations of the formula (4) provide approximations of the solution of system (3). They are even more accurate.

This representation allows development of new numerical schemes for the simulations of the solutions of system (3). For simplicity, consider the case where $E = \{1, \dots, n\}$. To approximate the solution over the interval $[0, \tau]$ define a time step Δt a number of points $T = \tau/\Delta t \in \mathbb{N}$ and replace \mathcal{U} and \mathcal{V} by the Toeplitz square matrices $\tilde{\mathcal{U}}$ and $\tilde{\mathcal{V}}$, generated by $i \in \{0, \dots, T-1\} \mapsto \int_{i\Delta t}^{(i+1)\Delta t} u(s) ds$, where u is the function generating \mathcal{U} (and similarly for \mathcal{V}). The number of operations needed is $\mathcal{O}((k+1)nT(n + \ln T))$ since the product with a Toeplitz matrix, as a convolution, has a cost $\mathcal{O}(T \ln T)$. This scheme has a first order accuracy, $\mathcal{O}(dt^\gamma + dx + \lambda^{k+1})$ where γ is equal to 1 for deterministic equations or $\frac{1}{2}$ if stochastic. In comparison, the Euler–Maruyama method has a complexity of $\mathcal{O}(T(n^2 + n \frac{\theta}{\Delta t} \ln(\frac{\theta}{\Delta t})))$ where θ is the support of g and an accuracy of $\mathcal{O}(dt^\gamma + dx)$, comparable to the expansion method in both aspects.

Two interesting advantages of the expansion over Euler-like methods are that (i) it is parallelizable (matrix, convolution products and the computation of the terms in the series can be done in parallel) and (ii) it appears to be numerically very stable, i.e. large Δt do not lead to a diverging scheme.

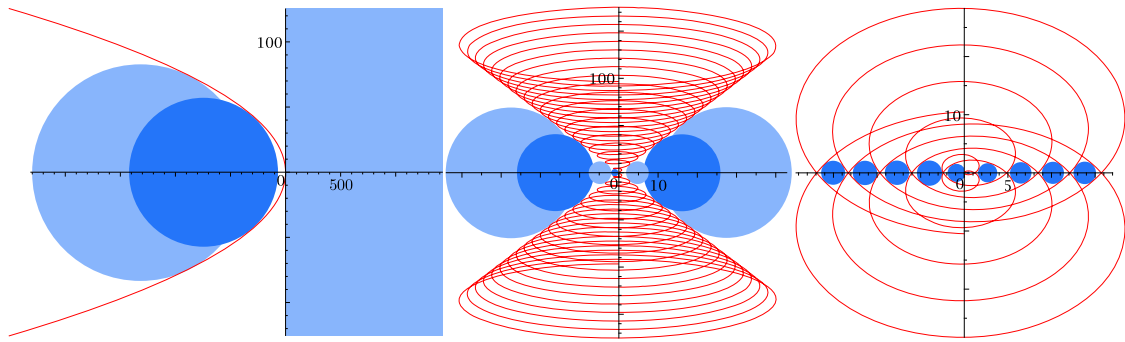


Fig. 1. Spectral constraint for different time-convolution kernels g : parametric plot of $\xi \in \mathbb{R} \mapsto \frac{2i\pi\xi}{\hat{g}(\xi)} \in \mathbb{C}$ (red), blue balls (of center $-l$ and of radius l) are the larger sets where the eigenvalues of a diagonalizable space operator A need to be contained for the expansion to be well-defined (eigenvalues need to belong to a single ball). To satisfy the spectral condition (1) the balls cannot intersect the red lines. (Left) Exponentially distributed delays with $\beta = 2\pi$ and $\xi \in [-20, 20]$. (Middle) Single delay with $\alpha = 2$ and $\theta = 1$ and $\xi \in [-20, 20]$. (Right) Single delay with $\alpha = 0.3$, $\theta = 1$ and $\xi \in [-5, 5]$.

3.3. Examples

Let us now treat some classical problems that are solved in the present framework and make explicit the choice of the parameter l .

Example 1 (Ornstein–Uhlenbeck process). The simplest example is the Ornstein–Uhlenbeck process with no delays (i.e. $g = \delta_0$). In that case, $\hat{g} = 1$, and therefore, for all $l \in \mathbb{R}$, $\inf_{\xi} \{|l + 2i\pi\xi|\} = |l|$ and the expansion is valid if there exists $l \in \mathbb{R}^*$ such that $\|l + A\| < |l|$, e.g. for any diagonalizable operator A whose spectrum is bounded and entirely contained in the left or right half plane. For negative matrices A (i.e. $l > 0$) $\mathcal{T}_h = \mathcal{U} = \mathcal{V}$ is a Toeplitz operator generated by the function $h : z \rightarrow e^{-lz}H(z)$ with H the Heaviside function. Therefore, the solution of $\dot{X} = A \cdot X + I + \Sigma \cdot dB$ can be written as $X_\tau = \sum_{k=0}^{+\infty} W^k \cdot (X_0\delta_0 + I_\tau + \Sigma \cdot dB_\tau) \cdot \mathcal{T}_h^{k+1}$. If $A \propto Id_C$ (e.g. in one dimension), the terms for $k > 0$ vanish in the above equation and we get the simple well-known expression $X_\tau = (X_0\delta_0 + I_\tau + dB_\tau) * h$. In the case where A is diagonalizable with real eigenvalues, it is easy to show analytically by a study of the function $l \mapsto \|l + A\|/|l|$ that the optimal choice for l is $(\mu_{\min} + \mu_{\max})/2$ where μ_{\min} and μ_{\max} are respectively the minimal and maximal eigenvalues of A . The case of complex eigenvalues can be treated analytically and algorithmically in the same manner.

Example 2 (Exponentially distributed delays). Let us now treat the case $g : x \mapsto \beta e^{-\beta x}H(x)$. In this case, $\hat{g}(\xi) = \frac{\beta}{\beta + 2i\pi\xi}$. Therefore, $\frac{2i\pi\xi}{\hat{g}(\xi)} = -2\pi(\frac{2\pi}{\beta}\xi^2 - i\xi)$ which corresponds to the red curve in the left picture of Fig. 1. Operators A satisfying the spectral condition (1) are the ones whose spectrum is contained in an open ball centered at $-l$ that does not intersect the red curve (blue disks in Fig. 1).

When A is negative, the operators \mathcal{U} and \mathcal{V} can be made completely explicit. Indeed, observing that: $\frac{\hat{g}(\xi)}{i\hat{g}(\xi) + 2i\pi\xi} = \frac{\beta}{(\frac{\beta}{2} + 2i\pi\xi)^2 - \frac{\beta^2\Delta^2}{4}} = \frac{\beta}{(\beta\frac{1+\Delta}{2} + 2i\pi\xi)(\beta\frac{1-\Delta}{2} + 2i\pi\xi)}$ with $\Delta = \sqrt{1 - 4l/\beta}$, the operator \mathcal{V} is the convolution operators generated by $\beta(h_- * h_+)$ with $h_{\pm} : t \mapsto e^{-\beta\frac{1\pm\Delta}{2}t}H(t)$. Similarly \mathcal{U} is generated by $\beta(h_- * h_+ + \frac{1}{\beta}h'_- * h_+)$. Even more explicitly, for $\beta > 4l$, \mathcal{V} is generated by $t \mapsto \frac{2}{\Delta}e^{-\frac{\beta}{2}t}sh(\frac{\beta\Delta}{2}t)$ and \mathcal{U} by $t \mapsto \frac{1}{\Delta}e^{-\frac{\beta}{2}t}(sh(\frac{\beta\Delta}{2}t) + \Delta ch(\frac{\beta\Delta}{2}t))$. When $4l > \beta$, a similar result holds replacing the hyperbolic functions ch and sh by \cos and \sin .

Example 3 (Single fixed delay). For $g = \delta_0 + \alpha\delta_\theta$, we have $\hat{g}(\xi) = 1 + \alpha e^{2i\pi\theta\xi}$. The convergence domain of the expansion (condition (1)) is shown in the middle and right pictures of Fig. 1, for two different $\alpha \in \mathbb{R}_+$. The red curve seems to be living on the two-dimensional projection of a simple three-dimensional cone of revolution whose section is a circle. In that case, it appears quite difficult to express \mathcal{U} and \mathcal{V} in a simple form, though their Fourier transform is explicit.

Remark. As illustrated in the previous example, a procedure to find the constant l such that the expansion converges consists in plotting on the same figure the complex eigenvalues of the spatial operator and the red curve $\xi \in \mathbb{R} \mapsto \frac{2i\pi\xi}{\hat{g}(\xi)} \in \mathbb{C}$ related to the time operators. If there exists a ball centered on the real axis which contains all the eigenvalues and that does not intersect the red curve, then choosing $-l$ as the value of its center ensures that the expansion will converge.

3.4. Stochastic heat equation

Let us now deal with a classical stochastic heat equation on \mathbb{S}^1 (described as the interval $[0, 1]$ where 0 and 1 are identified) as a classical example of linear partial differential equations:

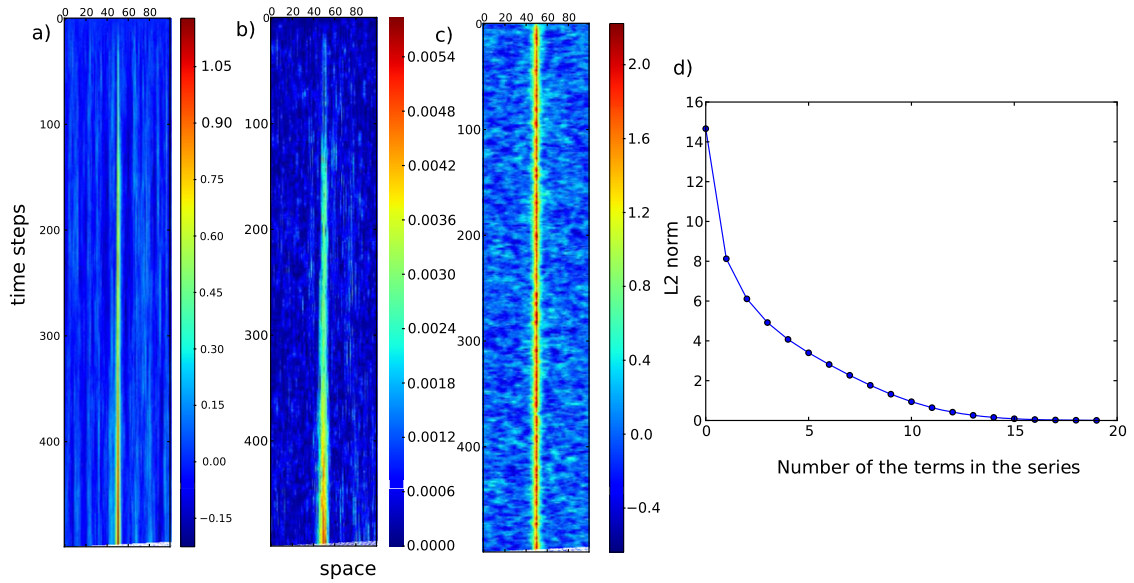


Fig. 2. Application of the expansion method to the stochastic heat equation on the circle with a Dirac source on the neuron in the middle. (a) Space-time diagram of the solution given by the expansion method for $dt = 0.01$. (b) Space-time diagram of the error between the solution in (a) and the solution given by Euler's method. (c) Space-time diagram of the solution given by the expansion method for $dt = 1$. (d) L_2 norm of the terms in the expansion. The parameters for these simulations are $n = 100$, number of time steps = 500, $\sigma = 0.1$ and $l = 2$.

$$\frac{\partial u}{\partial t}(x, t) = \Delta u(x, t) + v(x, t) + \sigma \eta(x, t)$$

with periodic boundary conditions, where Δ is the Laplacian on \mathbb{S}^1 , v is an external forcing and η is a multidimensional white noise. The input $v(x, t)$ is set to $\delta_{x=x_0}(x)$ and we take the initial condition $u(\cdot, t=0) = 0$.

The Laplacian operator has eigenvalues $-4\pi^2 k^2$ with $k \in \mathbb{N}$, corresponding to the eigenvectors $\cos(2k\pi x)$ and $\sin(2k\pi x)$. Since the eigenvalues are not bounded, it is not possible to find a suitable constant l to define the solution of the heat equation in our framework. However, the semi-discretized in space equation overcomes this problem by preventing the existence of very fast oscillations (corresponding to large eigenvalues of the Laplacian). We choose to discretize the space with N points regularly spaced, corresponding to a discretization step $dx = 1/N$. The resulting equation corresponds to (3) in dimension N , with $g = \delta(t)$ and $A \in \mathbb{R}^{N \times N}$ such that $A_{ij} = -2/dx^2$, $A_{ij} = 1/dx^2$ if $i = j \pm 1$, $A_{1n} = A_{n1} = 1/dx^2$ accounting for the periodicity of the medium, and $A_{ij} = 0$ otherwise. This matrix has eigenvalues in $[-4/dx^2, 0]$. This suggests the choice $l = 2/dx^2$ so that all the eigenvalues are in this ball a center $-l$ and radius l . This ball intersect the imaginary axis only in 0 (corresponding to spatially constant functions), so convergence issues might arise if one of the terms $W^k \cdot (\tilde{I}_\tau + \Sigma \cdot dB_\tau) \cdot \mathcal{U} \cdot \mathcal{V}^k$ is spatially constant, which clearly never occurs in our case. Therefore, our expansion is well-posed and provides a numerical scheme to compute the solution as shown in Fig. 2(d). In that figure, we exhibit the fact that the solution is well retrieved by the expansion, and the error compared to Euler's scheme with a time step $dt = 0.01$ (Fig. 2(b)) is more than two order of magnitude smaller than the solution. An interesting point of this method is that it works for any time step interval dt which is not the case for the Euler-Maruyama method which rapidly diverges as soon as the CFL condition is not satisfied for instance.¹ Moreover, extending the approach to a delayed formalism $g \neq \delta$ is costless in our framework.

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¹ For deterministic equations, stable methods such as implicit Euler schemes are stable. They require to invert a non-linear function and are hard to use in the stochastic case.