Partial Differential Equations/Mathematical Physics

Dissipation anomaly and energy cascade in 3D incompressible flows

Dissipation anormale et cascade énergétique pour des fluides incompressibles tridimensionnels

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ARTICLE INFO

ABSTRACT

The purpose of this Note is to present a mathematical evidence of the *dissipation anomaly* in 3D turbulent flows within a general setting for the study of energy cascade in *physical scales* of 3D incompressible flows recently introduced by the authors.

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RÉSUMÉ

Le but de cette Note est de présenter une mise en évidence mathématique de la *dissipation anormale* pour des flots turbulents tridimensionnels, dans un cadre général récemment introduit par les auteurs pour l'étude de la cascade énergétique aux *échelles physiques* dans les fluides incompressibles en trois dimensions.

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1. Introduction

Dissipation anomaly, i.e., non-vanishing of averaged energy dissipation rate in the infinite Reynolds number limit (sometimes referred to as 'zeroth law of turbulence'), is a key player in both empirical and phenomenological turbulence. On one hand, documented empirical confirmation dates back at least to Dryden's experiments on decaying turbulence in wind tunnels reported in 1943, and on the other hand, dissipation anomaly was a key postulate in Kolmogorov K41 phenomenology [11] (see also a discussion in [10,7,13]). K41 predicts existence of *energy cascade* – a local (in scale) nonlinear transfer of averaged energy across a range of scales, the so-called inertial range. In addition, the transfer rate/averaged energy flux is expected to be nearly-constant, and equal to averaged energy dissipation rate. Consequently, dissipation anomaly can be recast in terms of averaged energy flux as non-vanishing of averaged energy flux in the infinite Reynolds number limit (across a suitable range of scales).

Onsager in 1945–1949 [12,8] conjectured that irregular/singular solutions to the 3D Euler equations (the inviscid model) are capable of effectively dissipating all the energy in the flow. More precisely, Onsager conjectured that a minimal spatial regularity of a (weak) solution to the 3D Euler needed to conserve the energy is of the order of $(\frac{1}{3})^+$ (the main mathematical works confirming Onsager's $\frac{1}{3}$ criticality conjecture are [6,2,5,1]), and that in the case the energy is not conserved, the dissipation due to the singularities – *anomalous dissipation* – triggers the energy cascade which then continues indefinitely (unlike in the viscous case, the inertial range is expected to extend all the way to the zero-scale). This offers a way of

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studying dissipation anomaly in terms of vanishing viscosity limits of solutions to the 3D Navier–Stokes equations (NSE) converging to a singular solution to the 3D Euler exhibiting anomalous dissipation.

In a recent work, the authors introduced a setting for the study of energy cascade in 3D viscous incompressible flows in *physical scales*. (The preexisting work on existence of 3D energy cascade [9] took place in the Fourier space.) This led to a proof – under a suitable condition plausible in the regions of intense fluid activity – of both existence and scale-locality of the energy cascade in physical scales in decaying turbulence directly from the 3D NSE [3]. The general mathematical setting is the one of weak solutions satisfying the *local energy inequality*, and the key ingredient in the proof is suitable *ensemble averaging* of the local energy inequality. This method was then adopted to the 3D inviscid flows to show – under an assumption that the anomalous dissipation in the region of interest is strong enough (with respect to the energy) – both existence and scale-locality of the energy cascade in the 3D Euler flows extending *ad infinitum* confirming Onsager's predictions [4]. (One should note that the inviscid result is obtained in the setting of (at this point in time) *hypothetical* weak solutions to the 3D Euler satisfying the local energy inequality; a natural regularity here is L^3-locally in both space and time.)

The aim of this Note is to briefly review the aforementioned results – providing a unified approach to both viscous and inviscid cases – and observe that the inviscid result leads to a short proof of dissipation anomaly in this setting. Consider an L^3-strong in the space–time vanishing viscosity limit of weak solutions to the 3D NSE satisfying the local energy inequality converging to a (hypothetical) L^3 in the space–time weak solution to the 3D Euler. Duchon and Robert [5] noticed that the Euler solution then necessarily satisfies the local energy inequality. In fact, their calculation implies, assuming that the NSE solutions do not exhibit anomalous dissipation *per se*, that the local spatiotemporal viscous energy dissipation rates converge to the local spatiotemporal anomalous dissipation corresponding to the vanishing viscosity Euler solution, resulting in a local spatiotemporal dissipation anomaly (provided the inviscid anomalous dissipation is strictly positive). Here, we show that our approach to the study of energy cascade in physical scales of 3D incompressible flows leads to a *scale-to-scale* manifestation of spatiotemporal dissipation anomaly. More precisely, denoting the macro-scale in the problem by R_0, it is shown – provided the condition for the inviscid cascade holds – that for every R^*, $0 < R^* < R_0$, there exists ν^* positive, such that for any ν, $0 < \nu < \nu^*$, the averaged energy flux associated with the viscous solution u^ν is nearly-constant and comparable to the anomalous dissipation *throughout the inertial range* delineated by $[R^*, R_0]$. Moreover, the sufficient condition for the never-ending anomalous energy cascade in the inviscid case triggers the sufficient conditions for the energy cascade in the vanishing viscosity sequence with ever-expanding inertial ranges.

2. 3D viscous energy cascade revisited

Let $\mathbf{x}_0$ be in $B(\mathbf{0}, R_0)$ (R_0 being a given arbitrary length – interpreted as a macro-scale, such that $B(\mathbf{0}, 2R_0)$ is contained in Ω where Ω is the global spatial domain) and $0 < R \leqslant R_0$. For the exact localization procedure at scale R, around the point $\mathbf{x}_0$ – as well as the precise interpretation of the local (kinetic) energy transfer in this setting – the reader is referred to [3,4].

Let $\mathbf{u}$ be a weak solution to the 3D NSE on $\Omega \times (0, 2T)$ satisfying the *local energy inequality*,

$$\nu \iint |\nabla \mathbf{u}|^2 \phi \leqslant \iint \frac{1}{2} |\mathbf{u}|^2 (\partial_t \phi + \nu \triangle \phi) + \iint \left(\frac{1}{2} |\mathbf{u}|^2 + p \right) \mathbf{u} \cdot \nabla \phi, \tag{1}$$

for any non-negative test function ϕ (e.g., a *suitable weak solution*). Physically, the term $\nu \iint |\nabla \mathbf{u}|^2 \phi$ represents the local spatiotemporal energy dissipation rate due to viscosity, while a (non-negative) defect in the local energy inequality due to possible singularities/lack of smoothness can be interpreted as the local spatiotemporal anomalous dissipation. For a refined cut-off function $\phi = \phi_{\mathbf{x}_0, R, T}$ defined on $B(\mathbf{x}_0, 2R) \times (0, 2T)$ denote by $\Phi_{\mathbf{x}_0, R}$ a local *inward* flux,

$$\Phi_{\mathbf{x}_0, R}(t) = -\int \left((\mathbf{u} \cdot \nabla) \mathbf{u} + \nabla p \right) \cdot \mathbf{u} \phi \, d\mathbf{x} = \int \left(\frac{1}{2} |\mathbf{u}|^2 + p \right) \mathbf{u} \cdot \nabla \phi \, d\mathbf{x}, \tag{2}$$

and by $\hat{\Phi}_{\mathbf{x}_0, R}$ a local time-averaged flux per unit mass, $\hat{\Phi}_{\mathbf{x}_0, R, T} = \frac{1}{TR^3} \int \Phi_{\mathbf{x}_0, R}(t)$. Similarly, denote by $\hat{e}_{\mathbf{x}_0, R}$ a local time-averaged energy and by $\hat{\varepsilon}_{\mathbf{x}_0, R}$ a local time-averaged total (viscosity plus anomalous) energy dissipation rate, all per unit mass,

$$\hat{e}_{\mathbf{x}_0, R, T} = \frac{1}{TR^3} \iint \frac{1}{2} |\mathbf{u}|^2 \phi^{2\delta - 1}, \qquad \hat{\varepsilon}_{\mathbf{x}_0, R, T} = \frac{1}{TR^3} \iint \frac{1}{2} |\mathbf{u}|^2 (\partial_t \phi + \nu \triangle \phi) + \hat{\Phi}_{\mathbf{x}_0, R, T} \tag{3}$$

(for a suitable δ, $\frac{1}{2} < \delta \leqslant 1$).

Also denote by e and ε the time-averaged energy and total energy dissipation rate per unit mass associated to the macro-scale domain on $(0, 2T)$,

$$e = \hat{e}_{\mathbf{0}, R_0, T} \quad \text{and} \quad \varepsilon = \hat{\varepsilon}_{\mathbf{0}, R_0, T}. \tag{4}$$

In order to connect the inviscid case with the vanishing viscosity limit, we are adopting a slightly different approach from the one presented in [3] where the defect in the local energy inequality (1) was interpreted as the anomalous energy

flux due to singularities, while the time interval T was bounded below by R_0^2/ν. Here, besides considering total (viscous plus anomalous) energy dissipation rate, we will be keeping the time scale T independent of ν. However, the calculations are completely analogous to the ones presented in [3]. In particular, the following version of Theorem 4.1 in [3] holds. For the precise definition of the ensemble averaging process $\{\langle \cdot \rangle_R\}_{0 < R \leqslant R_0}$ – with respect to '(K_1, K_2)-covers at scale R', the reader is referred to [4]; here, we briefly present the main idea.

Let K_1 and K_2 be two positive integers. A cover $\{B(x_i, R)\}_{i=1}^n$ of $B(0, R_0)$ is a (K_1, K_2)-cover at scale R if $(\frac{R_0}{R})^3 \leqslant n \leqslant K_1(\frac{R_0}{R})^3$, and any point x in $B(0, R_0)$ is covered by at most K_2 balls $B(x_i, 2R)$. K_1 and K_2 represent global and local *multiplicities*, respectively. For any physical density of interest θ, consider time-averaged, per unit mass – spatially localized to the cover elements $B(x_i, R)$ – local quantities $\hat{\theta}_{x_i,R}$, $\hat{\theta}_{x_i,R} = \frac{1}{TR^3} \iint \theta \phi_i^\rho$ (for a suitable ρ, $0 < \rho \leqslant 1$), and denote by $\langle \Theta \rangle_R$ the ensemble average given by $\langle \Theta \rangle_R = \frac{1}{n} \sum_{i=1}^n \hat{\theta}_{x_i,R}$. A key observation is that $\langle \Theta \rangle_R$ being *stable* – nearly-independent on a particular choice of the cover (with the fixed multiplicities K_1 and K_2) – indicates there are no significant fluctuations of the sign of the density of θ at scales comparable or greater than R. Consequently, for an *a priori* sign-varying density (e.g., the flux density), the ensemble averaging process acts as a *coarse detector* of the *sign-fluctuations at scale R*.

Theorem 2.1. *Let τ be a modified Taylor length scale defined by $\tau_T = [(\frac{R_0^2}{T} + \nu)\frac{e}{\varepsilon}]^{1/2}$. There exist positive constants c and K – depending only on the cover parameters K_1 and K_2 – such that for any γ in $(0, 1)$, the condition $\tau_T^2 < \frac{\gamma}{c} R_0^2$ implies*

$$\frac{1}{K}(1 - \gamma)\varepsilon \leqslant \langle \Phi \rangle_R \leqslant K(1 + \gamma)\varepsilon \tag{5}$$

for all R inside the inertial range determined by $\left[\left(\frac{\tau_T^2}{\frac{\gamma}{c}(R_0^2 + \nu T) - \tau_T^2}\nu T\right)^{1/2}, R_0\right]$.

3. 3D inviscid energy cascade

In this section, we recall a sufficient condition for energy cascade in 3D inviscid flows obtained in [4], Section 3. Let **u** be a (hypothetical) weak L^3 in the space–time solution to the 3D Euler satisfying the local energy inequality,

$$\iint \frac{1}{2}|\mathbf{u}|^2 \partial_t \phi + \iint \left(\frac{1}{2}|\mathbf{u}|^2 + p\right)\mathbf{u} \cdot \nabla\phi = \varepsilon(\mathbf{u}; \phi) \geqslant 0 \tag{6}$$

$(\varepsilon(\mathbf{u}; \phi)$ is the anomalous dissipation associated to the support of ϕ). Keeping the notation in line with the previous section – focusing on $B(\mathbf{x}_0, R)$ – denote by $\hat{\epsilon}_{\mathbf{x}_0, R, T}$ a local time-averaged anomalous dissipation per unit mass associated to $B(\mathbf{x}_0, R)$, and define $\langle \Phi \rangle_R$, e, and ε as in the viscous case, setting $\nu = 0$. In the same spirit, define the anomalous Taylor scale τ_T by $\tau_T = (\frac{R_0^2}{T}\frac{e}{\varepsilon})^{1/2}$. The main result in [4] states the following.

Theorem 3.1. *There exist positive constants c and K – depending only on the cover parameters K_1 and K_2 – such that for any γ in $(0, 1)$, the condition $\tau_T^2 < \frac{\gamma}{c} R_0^2$ implies*

$$\frac{1}{K}(1 - \gamma)\varepsilon \leqslant \langle \Phi \rangle_R \leqslant K(1 + \gamma)\varepsilon \tag{7}$$

for all R inside the inertial range determined by $(0, R_0)$.

Note that in the inviscid case, once the energy cascade commences, it continues indefinitely towards the zero-scale as predicted by Onsager.

4. Dissipation anomaly and energy cascade

Let $\{\mathbf{u}^\nu\}$ be a family of weak solutions to the 3D NSE satisfying the local energy inequality converging strongly in $L^3(\mathbb{R}^3 \times (0, 2T))^3$ to a (hypothetical) weak L^3 solution to the 3D Euler **u**. As mentioned in the Introduction, Duchon and Robert [5] noticed that the inviscid limit **u** in this setting also satisfies the local energy inequality (6). A superscript ν will be used to denote the energy, the flux and the (total) energy dissipation rate corresponding to a solution $\mathbf{u}^\nu$, while the quantities related to **u** will be denoted as in the previous section. A straightforward computation – utilizing the identity $p = -\mathcal{R}_j \mathcal{R}_k(u^j u^k)$ on $\mathbb{R}^3$ ($\mathcal{R}_l$ denoting the l-th Riesz transform; $\mathbf{u} = (u^1, u^2, u^3)$) and the bound $|\nabla \phi_{\mathbf{x}, R, t}| \leqslant C_0/R$ – implies the following estimate; this is simply a quantitative version of the corresponding calculation in [5] written in our setting.

Lemma 4.1. *Let $\mathbf{x} \in \mathbb{R}^3$, $R > 0$ and $t \in (0, T]$. Then,*

$$\left|\Phi_{\mathbf{x}, R, t}^\nu - \Phi_{\mathbf{x}, R, t}\right| \leqslant c \frac{1}{tR^4} \|\mathbf{u}^\nu - \mathbf{u}\|_{L^3(\mathbb{R}^3 \times (0, 2T))^3}^3. \tag{8}$$

Theorem 4.1. *Suppose that the inviscid cascade condition, $\tau_T < \sqrt{\gamma/c}\,R_0$, holds for the inviscid limit* **u**. *Then, for every R^*, $0 < R^* < R_0$, there exists ν^* positive, such that for any ν, $0 < \nu \leqslant \nu^*$,*

$$\frac{1}{K_\gamma}\varepsilon \leqslant \langle \Phi^\nu \rangle_R \leqslant K_\gamma \varepsilon \tag{9}$$

throughout the inertial range determined by $[R^*, R_0]$, where $K_\gamma = \frac{K}{1-\gamma}$.

Proof. Note that the inviscid cascade condition allows us to apply Theorem 3.1 with $\gamma := \gamma + \delta$, $0 < \delta < c\tau_T^2/R_0^2 - \gamma$, obtaining

$$\frac{1}{K}(1 - \gamma - \delta)\varepsilon \leqslant \langle \Phi \rangle_R \leqslant \frac{K}{1-\gamma-\delta}\varepsilon$$

for any R, $0 < R \leqslant R_0$. On the other hand, (8) implies that for a fixed (K_1, K_2)-cover

$$\langle \Phi^\nu \rangle_R \to \langle \Phi \rangle_R, \quad \text{as } \nu \to 0$$

uniformly on $[R^*, R_0]$ for any R^*, $0 < R^* < R_0$. Combining the two estimates finishes the proof. □

Remark 4.1. In fact, it is plain to verify that for all ν, $0 < \nu < \nu^*$, the energy cascade

$$\frac{1}{K}(1 - \gamma)\,\varepsilon^\nu \leqslant \langle \Phi^\nu \rangle_R \leqslant K(1 + \gamma)\varepsilon^\nu$$

holds inside the ever-expanding inertial ranges $\left[\left(\frac{(\tau_T^\nu)^2}{\frac{\gamma}{c}(R_0^2+\nu T)-(\tau_T^\nu)^2}\nu T\right)^{1/2}, R_0\right]$. This provides an *intrinsic version of Theorem* 4.1 – without an explicit reference to the vanishing viscosity limit (of course, $\varepsilon^\nu \to \varepsilon$).

Acknowledgements

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