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Conformal dimension and combinatorial modulus of compact metric spaces

Dimension conforme et module combinatoire des espaces métriques compacts

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ABSTRACT

In this Note we study the Ahlfors regular conformal dimension ($\dim_{AR} X$) of a compact metric space X. This is a quasisymmetric numerical invariant, introduced by P. Pansu. It plays nowadays an important role in geometric group theory and in conformal dynamics. We show how to compute $\dim_{AR} X$ using a critical exponent Q associated to the combinatorial modulus. As a consequence of the equality $\dim_{AR} X = Q$, we obtain a general criterion ensuring that the AR conformal dimension is 1. The conditions are stated in terms of local cut points of X. Finally, we give applications of these results to the boundaries of Gromov hyperbolic groups and to the Julia sets of semi-hyperbolic rational maps.

RÉSUMÉ

L'objet principal de cette Note est l'étude de la dimension conforme Ahlfors régulière ($\dim_{AR} X$) d'un espace métrique compact X. C'est un invariant numérique de quasisymétrie, introduit par P. Pansu. Elle joue actuellement un rôle important en théorie géométrique des groupes et en dynamique conforme. On montre comment calculer $\dim_{AR} X$ à partir de modules combinatoires en considérant un exposant critique Q. Comme conséquence de l'égalité $\dim_{AR} X = Q$, on obtient un critère général de dimension un. Les conditions sont données en termes de points de coupure locale de X. On donne par ailleurs des applications de ces résultats aux bords des groupes hyperboliques et aux ensembles de Julia des fractions rationnelles semi-hyperboliques.

Version française abrégée

Un homéomorphisme $h:(X,d)\to(X',d')$ entre deux espaces métriques quelconques est dit *quasisymétrique*, s'il existe un homéomorphisme croissant $\eta:\mathbb{R}_+\to\mathbb{R}_+$ tel que :

$$\frac{d'(h(x),h(a))}{d'(h(y),h(a))}\leqslant\eta\left(\frac{d(x,a)}{d(y,a)}\right),$$

pour tous $x,y,a\in X$ avec $y\neq a$. Deux métriques d et d' sur X sont dites quasisymétriques (notation : $d\sim_{qs}d'$) si l'application identité $id:(X,d)\to(X,d')$ est quasisymétrique.

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Une métrique sur X est dite *Ahlfors régulière de dimension* α s'il existe une mesure de Radon μ sur X telle que la mesure de toute boule B_r de rayon $r \leqslant \operatorname{diam} X$ est comparable à une constante multiplicative près à r^α. Dans ce cas, μ est comparable à la mesure de Hausdorff α-dimensionnelle et la dimension de Hausdorff de X est égale à α.

Soit (X, d) un espace métrique compact, la *jauge conforme* de la métrique d sur X est définie par $\mathcal{J}(X, d) := \{d' \text{ métrique sur } X\colon d' \sim_{qs} d\}$. La dimension conforme d'un espace métrique est un invariant numérique par quasisymétrie, introduit par P. Pansu [15]. Elle joue actuellement un rôle important en théorie géométrique des groupes et en dynamique conforme. Dans cette Note on s'intéresse à la dimension conforme Ahlfors-régulière (AR), introduite par M. Bourdon et H. Pajot [4] :

$$\dim_{AR}(X, d) := \inf\bigl\{\alpha > 0\colon \exists d' \in \mathcal{J}(X, d) \text{ AR de dimension } \alpha\bigr\}. \tag{1}$$

La discrétisation a été une des approches importantes pour étudier la jauge conforme d'un espace métrique. Soient $a > 1$ et $\lambda > 1$ des constantes assez grandes. On considère une suite croissante $\{X_n\}_n$ d'ensembles a^{-n}-séparés maximaux à laquelle on associe les recouvrements $\mathcal{S}_n$ par les boules $B(x, \lambda a^{-n})$ associées aux éléments $x \in X_n$. On note $\mathcal{S} = \bigcup \mathcal{S}_n$.

A cette suite de recouvrements on associe un graphe métrique hyperbolique Z_d, dont le bord est quasisymétriquement équivalent à X, en adaptant une construction de Elek–Bourdon–Pajot [10,4]. Le graphe Z_d est le nerf des $\{\mathcal{S}_n\}_n$, i.e. les sommets de Z_d sont les boules de $\mathcal{S}$ et il y a une arête entre deux boules B et B' si $B \in \mathcal{S}_n$, $B' \in \mathcal{S}_m$, $|n - m| \leqslant 1$ et $B \cap B' \neq \emptyset$. En particulier, la jauge conforme de X coïncide avec la jauge conforme canonique du bord de Z_d. Ceci permet d'utiliser les techniques de la géométrie hyperbolique pour étudier la jauge conforme de X.

Le module combinatoire est une version discrète du module conforme de l'analyse complexe, mais qui, contrairement à ce dernier, est indépendant de toute structure analytique. Il est défini de la façon suivante. Pour chaque n on note G_n le nerf du recouvrement $\mathcal{S}_n$ (le sous-graphe de Z_d qui contient $\mathcal{S}_n$ et les arêtes qui joignent les boules de $\mathcal{S}_n$). Soient $n, k \geqslant 1$; pour chaque boule $B = B(x, \lambda a^{-k}) \in \mathcal{S}_k$, on considère la famille des chemins $\Gamma_n(B)$ dans G_{k+n} qui «joignent» la boule B avec le complémentaire de la boule $2B$; i.e. les chemins $\gamma = \{B_i\}_{i=1}^N$ de G_{k+n} tels que le centre de B_1 appartient à $B(x, a^{-k})$ et celui de B_N appartient à $X \setminus B(x, 2a^{-k})$. Étant donné $p > 0$, on définit le p-module combinatoire de la couronne $(B, 2B)$ en posant $M_{p,n}(B) := \inf_\rho \sum_{B' \in \mathcal{S}_{k+n}} \rho(B')^p$, où l'infimum est pris sur toutes les fonctions de poids $\rho : \mathcal{S}_{k+n} \to \mathbb{R}_+$ qui sont $\Gamma_n(B)$-admissibles i.e., tout chemin $\gamma \in \Gamma_n(B)$ vérifie $\ell_\rho(\gamma) = \sum_i \rho(B_i) \geqslant 1$. On associe ainsi à chaque $p > 0$ une suite $\{M_{p,n}\}_n$, où $M_{p,n} := \sup_{B \in \mathcal{S}} M_{p,n}(B)$ est le p-module combinatoire de X à l'échelle n. On s'intéresse au comportement asymptotique de la suite $\{M_{p,n}\}_n$ et à sa dépendance en p. A p fixé la suite $\{M_{p,n}\}_n$ vérifie une inégalité sous-multiplicative. Il est donc naturel de considérer *l'exposant critique* du module combinatoire défini par $Q := \inf\{p > 0\colon \liminf_n M_{p,n} = 0\}$. Le résultat principal de cette Note est le suivant :

Théorème 0.1. *Soit (X, d) un espace métrique compact tel que $\mathcal{J}_{AR}(X, d) \neq \emptyset$. Alors pour $a > 1$ et $\lambda > 1$ suffisamment grands, qui dépendent seulement de la géométrie de d, on a $Q = \dim_{AR}(X, d)$.*

La preuve est basée sur une description combinatoire de la jauge conforme AR. Cela confirme que la combinatoire du graphe Z_d contient toute l'information sur la jauge conforme AR de (X, d). Signalons que le résultat est vrai quelle que soit la topologie de X dès que $\mathcal{J}_{AR}(X, d)$ est non vide.

Le Théorème 0.1 permet de donner un critère général de dimension conforme AR un. On rappelle que (X, d) est dit linéairement connexe s'il existe une constante $C \geqslant 1$ telle que pour tous $x, y \in X$ il existe une courbe γ qui les joint et de diamètre majoré par $Cd(x, y)$.

Théorème 0.2. *Soit (X, d) un espace métrique compact et linéairement connexe. Supposons qu'il existe une constante N telle que la condition suivante (appelée condition UBR) est vérifiée : pour toute boule $B(x, r)$ de X il existe un ensemble fini $P \subset B(x, 2r)$ de cardinal majoré par N, tel que toute courbe γ qui intersecte à la fois $B(x, r)$ et $X \setminus B(x, 2r)$ intersecte aussi P. Alors* $\dim_{AR} X = 1$.

La condition UBR, avec le critère de Mackay [14] de dimension conforme strictement plus grande que un, donnent une vision conceptuelle assez claire sur la relation entre la dimension conforme et les points de coupure locale.

Les Théorèmes 0.1 et 0.2 peuvent être appliqués pour donner une réponse partielle à la question de savoir quels sont les groupes hyperboliques dont le bord est de dimension conforme AR égale à un. On trouve aussi de nouveaux exemples d'espaces de dimension conforme AR égale à un parmi les ensembles de Julia des fractions rationnelles semi-hyperboliques, en particulier tous les polynômes semi-hyperboliques de Julia connexe.

1. Introduction

A homeomorphism $h : (X, d) \to (X', d')$ between two metric spaces is *quasisymmetric* if there exists an increasing homeomorphism $\eta : \mathbb{R}_+ \to \mathbb{R}_+$ so that for all $x, y, a \in X$, with $y \neq a$, we have

$$\frac{d'(h(x), h(a))}{d'(h(y), h(a))} \leqslant \eta\left(\frac{d(x, a)}{d(y, a)}\right).$$

In words, a homeomorphism is quasisymmetric if it distorts the relative distances between points in a uniform and scale invariant fashion. We say that two distances d and d' in X are quasisymmetrically equivalent (notation: $d \sim_{qs} d'$) if the identity map $id : (X, d) \to (X, d')$ is a quasisymmetric homeomorphism.

Let (X, d) be a compact metric space. Its quasisymmetric invariant properties are encoded within its *conformal gauge*:

$$\mathcal{J}(X,d) := \{d' \text{ distance on } X\colon\ d' \sim_{qs} d\}. \tag{2}$$

A distance $d' \in \mathcal{J}(X,d)$ is said to be Ahlfors regular (AR for short) of dimension $\alpha > 0$ if there exists a Radon measure μ on X and a constant $C \geqslant 1$ such that for any ball B_r of radius $0 < r \leqslant \operatorname{diam}_{d'} X$ we have

$$\frac{1}{C} \leqslant \frac{\mu(B_r)}{r^\alpha} \leqslant C. \tag{3}$$

In that case μ is comparable to the α-dimensional Hausdorff measure and $\alpha = \dim_H(X, d')$ is the Hausdorff dimension of (X, d'). See [13] for a detailed exposition of these notions.

In this Note we are interested in the AR *conformal dimension* of (X, d); it is a quasisymmetric invariant introduced by Pansu in [15]. In fact, the definition given here is a slight modification due to Bourdon and Pajot [4]. It is defined as follows

$$\dim_{AR}(X,d) := \inf\{\alpha > 0\colon\ \exists d' \in \mathcal{J}(X,d) \text{ AR of dimension } \alpha\}. \tag{4}$$

The interest in the study of quasisymmetric invariants comes from the strong relationship between the geometric properties of a (Gromov) hyperbolic space and the analytical properties of its boundary at infinity. Quasi-isometries between hyperbolic spaces induce quasisymmetric homeomorphisms between their boundaries, so any quasisymmetric invariant gives a quasi-isometric one. For example, the conformal dimension was introduced as a tool for classifying homogeneous spaces of negative curvature up to quasi-isometry [15]. For hyperbolic groups, the understanding of the canonical conformal gauge of the boundary at infinity (induced by the visual metrics) is an important step in the approach of Bonk and Kleiner to the problem of characterizing uniform lattices of $PSL_2(\mathbb{C})$ via their boundaries (Cannon's conjecture) [2]. They showed that Cannon's conjecture is equivalent to the following: if G is a hyperbolic group whose boundary is homeomorphic to the topological two-sphere S^2, then the AR conformal dimension of ∂G is attained. Motivated by Sullivan's dictionary, Haïssinsky and Pilgrim translated these notions to the context of branched coverings [12]. In particular, the AR conformal dimension characterizes rational maps between CXC branched coverings (see [12]).

The proofs of the results stated in Sections 2 and 3 are not given but can be found in [8].

2. Main results

Discretization of conformal analytical objects has proved to be a useful tool in the study of the conformal gauge of metric spaces. Versions of combinatorial modulus have been considered by several authors in connection with Cannon's conjecture (see [7,1,11]). The combinatorial modulus is a discrete version of the analytical conformal modulus from complex analysis, but unlike the latter, is independent of any analytical framework. It is defined using coverings of X and therefore depends only on the combinatorics of such coverings. In [3] the authors proved several important properties of combinatorial modulus for approximately self-similar sets. Here we define a combinatorial modulus of a metric space (X, d) that takes into account all the "annuli" of the space with some fixed radius ratio.

Let $a > 1$ and $\lambda > 1$ be two large enough constants. By a discretization of (X, d) we mean a sequence $\{X_n\}_n$ of maximal r_n-separated subsets of X where $r_n = a^{-n}$; that is for all $k \geqslant 1$ and $x \neq y \in X_n$ we have $d(x, y) \geqslant r_n$, and the set of balls $\mathcal{U}_n := \{B(x, r_n)\colon\ x \in X_n\}$ is an open covering of X. We write $\mathcal{S}_n := \{B(x, \lambda r_n)\colon\ x \in X_n\}$ and $\mathcal{S} := \bigcup_{n \geqslant 0} \mathcal{S}_n$.

In [4] (see also [10]) the authors construct a hyperbolic metric graph Z_d with boundary homeomorphic to X, and such that with this identification the distance d is a visual metric on ∂Z_d. The graph Z_d is the nerve of the sequence of coverings $\mathcal{S}$, i.e. the vertices are the balls in $\mathcal{S}$ and two balls B and B' are connected by an edge if $B \in \mathcal{S}_n$, $B \in \mathcal{S}_m$ with $|n - m| \leqslant 1$ and $B \cap B' \neq \emptyset$. For each $n \geqslant 0$ we denote by G_n the subgraph of Z_d which consists of all vertices in $\mathcal{S}_n$ and all the edges of Z_d that connects two of them.

Fix for the moment $n \geqslant 1$. For each $k \geqslant 0$ and each ball $B = B(x, \lambda r_k) \in \mathcal{S}_k$ we define $\Gamma_n(B)$ to be the set of paths γ with vertices $\{B_i\}_{i=1}^N$ of G_{k+n} such that the center of B_1 belongs to $B(x, r_k)$ and that of B_N belongs to $X \setminus B(x, 2r_k)$. We consider the set $A_n(B)$ of all admissible weight functions $\rho : \mathcal{S}_{k+n} \to \mathbb{R}_+$; i.e. $\forall \gamma \in \Gamma_n(B)$ we have

$$\ell_\rho(\gamma) := \sum_{i=1}^N \rho(B_i) \geqslant 1. \tag{5}$$

Let $p > 0$. We define the *p-combinatorial modulus* associated to the ball $B \in \mathcal{S}_k$ at scale n as

$$\operatorname{Mod}_p(B, n) := \inf_{\rho \in A_n(B)} \left\{ \sum_{B' \in \mathcal{S}_{k+n}} \rho(B')^p \right\}, \tag{6}$$

and the p-combinatorial modulus of X at scale n as

$$M_{p,n} := \sup_{B\in\mathcal{S}} \mathrm{Mod}_p(B,n). \tag{7}$$

We are interested in the asymptotic behavior of $M_{p,n}$ as n tends to infinity and its dependence on p. We set $M_p := \liminf_n M_{p,n}$. For fixed $p > 0$ the sequence $\{M_{p,n}\}_n$ verifies a sub-multiplicative inequality. This allows to define a *critical exponent* $Q := \inf\{p > 0{:}\ M_p = 0\}$. The main result of this Note is the following:

Theorem 2.1. *Let (X,d) be a compact metric space such that $\mathcal{J}(X,d) \neq \emptyset$. Then, for $a > 1$ and $\lambda > 1$ that depends only on the geometric properties of d, we have $Q = \dim_{AR}(X,d)$.*

Idea of the proof. Using the graph Z_d, we can give a combinatorial description of the AR conformal gauge of (X,d). From an appropriate weight function $\rho : \mathcal{S} \to (0,1)$, one can change the lengths of the edges of Z_d and obtain a metric graph Z_ρ quasi-isometric to Z_d. This graph admits a visual metric θ_ρ in $\mathcal{J}_A R(X,d)$ of controlled Hausdorff dimension. When ρ goes through all the possible choices, we get all the gauge $\mathcal{J}_{AR}(X,d)$ up to bi-Lipschitz homeomorphisms.

By definition, for $p > Q$, the combinatorial moduli $M_{p,n}$ tend to zero as n tends to infinity. So one can choose n large enough, depending on the difference $p - Q$, so that $\mathrm{Mod}_p(B,n)$ is small for all the balls $B \in \mathcal{S}$. This gives some flexibility to change the optimal weight functions, so as to obtain a weight function $\rho : \mathcal{S} \to (0,1)$ which satisfies the conditions of the combinatorial description of the gauge. This gives an AR metric θ_p in $\mathcal{J}(X,d)$ of dimension p. The distortion of $id : (X,d) \to (X,\theta_p)$ depends on n (and thus on the difference $p - Q$). □

This result confirms that the combinatorics of the graph Z_d contains all the information of the AR conformal gauge of X. It should be noted that it is true regardless of the topology of X, it just requires that $\mathcal{J}_{AR}(X,d)$ be non-empty.

Theorem 2.1 enables to give a general criterion to get AR conformal dimension one. We recall that (X,d) is *linearly connected* if there exists a constant $C \geqslant 1$ such that any two points $x, y \in X$ can be connected by a curve γ of X with $\operatorname{diam}\gamma \leqslant Cd(x,y)$.

Theorem 2.2. *Let X be a compact and linearly connected metric space. Suppose there exists a constant N so that the following condition (called UBR condition) is satisfied: for any ball $B(x,r)$ of X, there is a finite set $P \subset B(x,2r)$ of cardinality bounded by N such that any curve γ intersecting both $B(x,r)$ and $X \setminus B(x,2r)$ intersects also P. Then $\dim_{AR} X = 1$.*

The UBR condition and Mackay's criterion [14] give a fairly clear vision about the relationship between conformal dimension and local cut points. Bruce Kleiner informed me that with Stephen Keith he has similar unpublished results.

3. Applications

An interesting question in geometric group theory is to know which are the hyperbolic groups that have a boundary at infinity of AR conformal dimension one. Theorems 2.1 and 2.2 give a partial answer to this question. By the work of Dunwoody and Stallings [9,16] on the accessibility of finitely presented groups, there is an action of G on a simplicial tree Σ, without edge inversions and with finite quotient, such that the edge stabilizers are finite and the vertex stabilizers have at most one end. We call this action a DS splitting of G. We have the following result:

Corollary 3.1 *(Stability by splittings over finite groups). Let G be a hyperbolic group and let Σ be the tree associated to the DS splitting. We denote by $\{v_1,\ldots,v_M\}$ a set of representatives of the orbits of the vertices of Σ and $G(v_i)$ their respective stabilizers. Then*

- *the AR conformal dimension of G is zero if all $G(v_i)$, $i = 1,\ldots,M$, are finite, or*
- $\dim_{AR}\partial G = \max\{\dim_{AR}\partial G(v_i){:}\ G(v_i)$ *is infinite*$\}$ *otherwise.*

For a one-ended hyperbolic group G, after the work of Bowditch [5], we know that the structure of local cut points is described by its canonical JSJ decomposition. We have the following result:

Corollary 3.2. *Let G be a one-ended hyperbolic group. If G is a cocompact Fuchsian group, or there is no vertices of rigid type in the JSJ splittings of G, then ∂G satisfies the UBR condition. In particular* $\dim_{AR}\partial G = 1$.

Corollaries 3.1 and 3.2 are the analogues in the broader context of hyperbolic groups of results proved for Kleinian groups in [6].

Sullivan's dictionary also shows analogies between the AR conformal dimension of hyperbolic groups and of Julia sets of semi-hyperbolic rational maps. Let f be a semi-hyperbolic rational map with connected Julia set J and let P denote the (countable) set of periodic and preperiodic points. We say that two points $x, y \in J$ are in the same *fiber* if for every finite subset F of P the points x and y are in the same connected component of $J \setminus F$. The fibers are said to be trivial if every fiber is a singleton. In this case J satisfies the UBR condition and we have:

Corollary 3.3. *Let f be a semi-hyperbolic rational map with connected Julia set J and such that the fibers are trivial* (*this is the case e.g. of polynomials*)*. Then* $\dim_{AR} J = 1$.

This gives new examples of spaces of AR conformal dimension equal to one.

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